Elliptic Curves, Algebraic Geometry Approach in Gravity Theory and Some Applications in Theories with Extra Dimensions I.

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Dedicated to the memory of Prof. Sawa S. Manoff /1943 - 27.05. 2005/

Abstract

Motivated by the necessity to find exact solutions with the elliptic Weierstrass function of the Einstein's equations (see gr-qc/0105022),the present paper develops further the proposed approach in hep-th/0107231 ,concerning the s.c. cubic algebraic equation for effective parametrization. Obtaining an "embedded" sequence of cubic equations, it is shown that it is possible to parametrize also a multi-variable cubic curve, which is not the standardly known case from algebraic geometry. Algebraic solutions for the contravariant metric tensor components are derived and the parametrization is extended in respect to the covariant components as well.

It has been speculated that corrections to the extradimensional volume in theories with extra dimensions should be taken into account, due to the non-euclidean nature of the Lobachevsky space.It was shown that the mechanism of exponential "damping" of the physical mass in the higher-dimensional brane theory may be more complicated due to the variety of contravariant metric components for a spacetime with a given constant curvature. The invariance of the low-energy type I string theory effective action is considered in respect not only to the known procedure of compactification to a four-dimensional spacetime, but also in respect to rescaling the contravariant metric components. As a result, instead of the simple algebraic relations between the parameters in the string action, quasilinear differential equations in partial derivatives are obtained, which have been solved for the most simple case.

In the Appendix, a new block structure method is presented for solving the well known system of operator equations in gravity theory in the N-dimensional case.

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Contents

1 INTRODUCTION

Inhomogeneous cosmological models have been intensively studied in the past in reference to colliding gravitational waves [1] or singularity structure and generalizations of the Bondi

- Tolman and Eardley-Liang-Sachs metrics [2, 3]. In these models the inhomogeneous metric is assumed to be of the form [2]

$$ds^2 = dt^2 - e^{2\alpha(t,r,y,z)}dr^2 - e^{2\beta(t,r,y,z)}(dy^2 + dz^2) \tag{1.1}$$

(or with $r \rightarrow z$ and $z \rightarrow x$) with an energy-momentum tensor $G_{\mu\nu} = k\rho u_\mu u_\nu$ for the irrotational dust and functions $\alpha(t, r, y, z)$ and $\beta(t, r, y, z)$, determined by the Einstein's equations. The particular form of the metric and therefore of the functions α and β may restrict seriously the spacetime symmetries, due to which in some cases, for example when there is a two - parameter group of motions (i. e. $\beta = \beta(t, y)$) with commuting Killing vectors, then "there are no solutions to these equations" [2]. This represents in fact the serious motivation why the functions α and β are chosen in a special form [4]

$$\alpha \equiv log\left[h(z, \xi, \overline{\xi})\Phi' + h(z, \xi, \overline{\xi})\Phi\nu'\right] \quad , \tag{1.2}$$

$$\beta \equiv log\Phi(t, z) + \nu(z, \xi, \overline{\xi}) \quad , \tag{1.3}$$

so that the integrations of (some) of the components of the Einstein's equations is ensured. In (1.2) and (1.3) $' \equiv \frac{\partial}{\partial z}$, $. \equiv \frac{\partial}{\partial t}$ and $\xi, \overline{\xi}$ are a pair of complex (conjugated) variables

$$\xi \equiv x + iy \quad ; \quad \overline{\xi} \equiv x - iy \quad . \tag{1.4}$$

A nice feature of the approach is that in the limit $t \rightarrow \infty$ [5] and under a special choice of the pressure as a definite function of time the metric approaches an isotropic form [4]. Other papers, also following the approach of Szafron-Szekerez are [6,7]. In [7], after an integration of one of the components - G_1^0 of the Einstein's equations, a solution in terms of an elliptic function is obtained.

In different notations, but again in the framework of the Szafron-Szekerez approach the same integrated in [7] **nonlinear differential equation**

$$\left(\frac{\partial\Phi}{\partial t}\right)^2 = -K(z) + 2M(z)\Phi^{-1} + \frac{1}{3}\Lambda\Phi^2 \tag{1.5}$$

was obtained in the paper [8] of Kraniotis and Whitehouse. They make the useful observation that (1.5) in fact defines a (cubic) algebraic equation for an elliptic curve, which according to the standard algebraic geometry prescribtions (see [9] for an elementary, but comprehensive and contemporary introduction) can be parametrized with the **elliptic Weierstrass function**

$$\rho(z) = \frac{1}{z^2} + \sum_{\omega}\left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right] \tag{1.6}$$

and the summation is over the poles in the complex plane. More explicitely the simple idea about the parametrization procedure shall be explained in Sect. II of the present paper. However, two important problems immediately arise, which so far have remained without an answer:

1. The parametrization procedure with the elliptic Weierstrass function in algebraic geometry is adjusted for cubic algebraic equations with number coefficients! Unfortunately, equation (1.5) is not of this type, since it has coefficient functions in front of the variable Φ, which depend on the complex variable z. In view of this, it makes no sense to define "Weierstrass invariants" as

$$g_2 = \frac{K^2(z)}{12} \quad ; \quad g_3 = \frac{1}{216}K^3(z) - \frac{1}{12}\Lambda M^2(z) \quad , \tag{1.7}$$

since the above **functions** have to be set up equal to the complex numbers g_2 and g_3 (the s. c. Eisenstein series)

$$g_2 = 60\sum_{\omega\subset\Gamma}\frac{1}{\omega^4} = \sum_{n,m}\frac{1}{(n+m\tau)^4} \quad , \tag{1.8}$$

$$g_3 = 140\sum_{\omega\subset\Gamma}\frac{1}{\omega^6} = \sum_{n,m}\frac{1}{(n+m\tau)^6} \tag{1.9}$$

and therefore **additional equations** have to be satisfied in order to ensure the parametrization with the Weierstrass function.

2. Is the Szekerez - Szafron metric the only case, when the parametrization with the Weierstrass function is possible? Closely related to this problem is the following one - is only one of the components of the Einstein's equation parametrizable with $\rho(z)$ and its derivative?

An attempt to give an answer to the first problem was made in the preceeding paper [10], where it has been proved that the Weierstrass function and its derivative satisfy a "parametrizable" form of a cubic equation of a more general type

$$\left[\rho^{'}(z)\right]^2 = 4\rho^3(z) - g_2(z)\rho(z) - g_3(z) \tag{1.10}$$

with convergent coefficients (at $n \to \infty$) in the Laurent expansion of the functions $g_2(z)$ and $g_3(z)$. Note also that still the more general problem about the validity of the equation

$$(\frac{d\rho}{dz})^2 = M(z)\rho^3 + N(z)\rho^2 + P(z)\rho + E(z) \tag{1.11}$$

remains unsolved. In future applications, we shall follow the standard approach in parametrizing a cubic algebraic equation with number coefficients - the s.c."arithmetic theory".

Concerning the second problem, which is of primary importance in view of possible applications in multidimensional gravity theories, M-theory and supergravity theories, it will be treated in the second part of this paper. In it the developed previously in [10] parametrization procedure for the s. c. cubic algebraic equation of reparametrization invariance of the gravitational Lagrangian and as well the proposed in this (first) part method for parametrization of embedded sequence of cubic algebraic equations will be applied in respect to the Einstein's system of equations in their general form (i.e. not dependent on any concrete metric and also with an arbitrary energy - momentum tensor). The treatment will be performed within the assumption that the components of the metric tensor can be represented as a factorized product of two vector fields (but not necessarily lying in the tangent bundle of the given manifold), i. e.

$$g^{\alpha\beta}(\overrightarrow{x}) = k^{\alpha}(\overrightarrow{x})k^{\beta}(\overrightarrow{x}) \quad ; \quad \overrightarrow{x} = (x^1, x^2, ..., x^n) \qquad \alpha, \beta = 1, 2, ...n \quad . \tag{1.12}$$

Then the Einstein's equations, if written in respect to the $n-$th component k^n of the vector field $\overrightarrow{k} = (k^1, k^2, ..., k^n)$, can be represented as a system of n (multicomponent) **cubic** algebraic equations and additionally $\left(\begin{array}{c} n \\ 2 \end{array}\right) = \frac{n(n-1)}{2}$ **quartic** algebraic equations. Consequently, it is not surprising at all that parametrization with the Weierstrass function and its derivative can be performed in respect to some of the components in the Einstein's system of equations not only in the Szekeres - Szafron case, but also in the general case as well. Note also that the parametrization with the Weierstrass function can be applied also in respect to quartic algebraic equations [11, 12].

The first part of the present paper continues and develops further the approach from the previous paper [10], where a definite choice of the contravariant metric tensor was made in the form of the factorized product $\widetilde{g}^{ij} = dX^i dX^j$ and $X^i = X^i(x_1, x_2, x_3, ..., x_n)$ are generalized coordinates. They can be regarded as an $n-$ dimensional hypersurface, defining a transition from an **initially defined** set of coordinates $x_1, x_2, x_3, ..., x_n$ on a chosen manifold to another set of the generalized coordinates $X^1, X^2, ..., X^n$. In section 1I of the present paper it will be reminded briefly how the cubic algebraic equation in respect to the differentials dX^i was derived in [10], but in fact the aim will be to show that depending on the choice of variables in the gravitational lagrangian or in the Einstein's equations, a wide variety of algebraic equations (of third, fourth, fifth, ninth and tenth degree) in gravity theory may be treated, if a distinction between the covariant metric tensor components and the contravariant ones is made. This idea, originally set up by Schouten and Schmutzer, was further developed in the papers of S.Manoff, mainly with the

purpose of classification of such more general theories of gravity with two different metrics and affine connections (covariant and contravariant affine connections), or also theories, admitting torsion and shear (for a review of the general approach, one may consult the review article [13]). Further, possible observational consequences of propagation of signals in spaces with affine connections and metrics were explored in [14].

The present paper will also be related with the idea about the distinction between the covariant and contravariant metric components should be made, although no shear or torsion will be assumed to exist. The physical idea, which will be exploited will be: can such a gravitational theory with a more general contravariant tensor be equivalent to the usual and known to us theory with a contravariant metric tensor, which is at the same time the inverse one of the covariant one? By "equivalence" it is meant that the gravitational lagrangian in both approaches should be equal, on the base of which the s.c. cubic algebraic equation for reparametrization invariance (of the gravitational Lagrangian) was obtained in [10]. The derivation was based also on the construction of another connection $\widetilde{\Gamma}_{kl}^{s} \equiv \frac{1}{2}dX^{i}dX^{s}(g_{ik,l}+g_{il,k}-g_{kl,i})$, which **is the same as the known Christoffell connection Γ_{kl}^{s}, but with the contravariant tensor component replaced with the factorized product** $dX^{i}dX^{s}$. The connection $\widetilde{\Gamma}_{kl}^{s}$ has two very useful properties: 1. It has an affine transformation law, i.e. it transforms after change of coordinates as the usual Christoffell connection. For completeness the proof is given in Appendix A of the present paper, it was not given in the previous paper [10]. 2. $\widetilde{\Gamma}_{kl}^{s}$ is an **equiaffine connection** (see also Appendix A for the elementary proof), which is a typical notion, introduced in classical **affine geometry** [15, 16] and meaning that there exists a volume element, which is preserved under a parallel displacement of a basic $n-$dimensional vector $e \equiv e_{i_1 i_2 \ldots . i_n}$. However, this notion turns out to be very convenient and important, since for such types of connections we can use the known formulae for the Ricci tensor, but with the connection $\widetilde{\Gamma}_{kl}^{s}$ instead of the usual Christoffell one Γ_{kl}^{s} and moreover, the Ricci tensor $\widetilde{R}_{ij}$ will again be a symmetric one, i. e. $\widetilde{R}_{ij}=\widetilde{R}_{ji}=\partial_k\widetilde{\Gamma}_{ij}^{k}-\partial_i\widetilde{\Gamma}_{kj}^{k}+\widetilde{\Gamma}_{kl}^{k}\widetilde{\Gamma}_{ij}^{l}-\widetilde{\Gamma}_{ki}^{m}\widetilde{\Gamma}_{jm}^{k}$. It should be emphasized that the derivation of the cubic equation and of all the other algebraic equations in principle does not depend on the properties of the affine connection. Also, the theoretical reasoning for the distinction between covariant and contravariant components comes again from affine geometry, where the assumption about the existence of inverse contravariant metric components in fact "sets up" the correspondence between "covectors" (in our terminology - these are the "vectors") and the "vectors" (i.e. the contravariant vectors"), on which the covariant and the contravariant tensors are being built.. However, **such a correspondence is not necessarily to be established** (see again [15, 16]) and both kinds of tensors have to be treated as different mathematical objects, defined on one and the same manifold. For a contemporary introduction in affine geometry, one may consult also the monographs [17, 18].

Let us point out some important consequences from the distinction between covariant and contravariant metric components: 1. One can perform a classification of all the affine connections, resulting in the same gravitational Lagrangian. In a sense, this is a problem, which has been solved already in the 60-ties, but here the proposal is to perform this classification on the base of solving algebraic equations instead of nonlinear

differential equations in partial derivatives. 2. One can find all the contravariant metric tensor components, satisfying the same gravitational Lagrangian. The last fact brings some new and important consequences for gravitational physics in the sense that if one considers the Einstein's equations, one can speak about **separate classes of solutions for the covariant metric tensor components and for the contravariant ones.** These (i.e. both) components constitute the algebraic variety, satisfying the Einstein's equations, if **they are considered as a set of intersecting multivariable cubic and quartic algebraic surfaces (**further instead of cubic surfaces we shall continue to use the terminology "cubic curves"). But in principle, the notion of "intersecting algebraic curves and surfaces" in algebraic geometry is very complicated and unfortunately, even though some approaches may be found in older monographs (see for example vol. II of [19]), they are purely mathematical ones and not at all applicable to really given algebraic curves. The second volume of the known monograph [19] of Hodge and Pedoe contains some material about intersection of quadratic manifolds - these are simply quadratic algebraic equations, which for our concrete case are for example the equations from the quadratic system $g_{ij}\widetilde{g}^{jk} = g_{ij}dX^jdX^k = \delta_i^k$, determining the differentials dX^i from the defining equality $g_{ij}\widetilde{g}^{jk} = \delta_i^k$ for the existence of the inverse contravariant metric tensor components $\widetilde{g}^{jk}$. By itself, the knowledge about this system of equations is useful since it helps to make the clear distinction between a **linear algebraic system of equations** (where the unknown variables are in a vector-column) and an **operational system of equations (**see the monograph of Gantmacher [20] for treatment of such equations), in which the unknown variables constitute a matrix. The above mentioned system is definitely of the second type and not of the first type. This is an important point, since it may be thought that if the determinant $det \mid dX^idX^j \mid= 0$ (which really happens for any dimensions), then the system does not have a solution. It has been shown in Appendix C, and in part also in section 1I, on the base of an originally created approach with the help of the s.c. "block structure matrices" and for the general $N-$dimensional case, that yet the system is correctly determined in respect to g_{ij}.

But now, it is another point if we would like to treat the same system as a system of **quadratic algebraic equations**, intersecting the set of **cubic and quartic algebraic Einstein's equations**. Therefore, extending the theory by admitting more general contravariant metric tensor components and finding the intersection variety (i.e. the algebraic variety of the variables dX^i, satisfying both systems of equations), the **standardly known solutions of the Einstein's equations should be recovered**. In principle, the s.c. "intersection theory" in algebraic geometry is very complicated, and it is not very clear how a concrete complicated system of intersecting algebraic equations can be analyzed. In this paper we shall investigate **only** the cubic algebraic equation for reparametrization invariance of the gravitational Lagrangian. It may be noted that general theorems for intersection of algebraic curves of different (arbitrary) degrees are given in [21, 22], but they refer to the following cases: a) plane curves, b) a system of one - parametric curves of degree n, having a known quantity of common points with another curve of degree p. c) two intersecting curves of different degrees and a **third one,** passing through a **known number of the intersection points** of the first two curves and some

other similar cases. As it may be guessed, not a single of any these (and other) cases is adapted for the treatment of the intersecting systems of cubic, quartic and quadratic algebraic equations in gravity theory.

Yet one of the serious shortcomings of the algebraic geometry approach is related to the fact that the standardly known procedure for parametrization of a cubic algebraic equation with the Weierstrass function is adapted to 1. plane cubic algebraic curves of two variables. 2. algebraic curves with constant coefficients. We already commented at the beginning (and also in [10]) on the second point, so in this paper all efforts will be concentrated on the problem **how to parametrize a given non - plane, multicomponent cubic algebraic curve with the Weierstrass function and its derivative - in the present case, this is of course the equation of reparametrization invariance.** Previously [10], it has been shown how a selected variable dX^{i} of the cubic algebraic equation and the ratio $\frac{a}{c}$ of two coefficient functions in the performed linear - fractional transformation can be parametrized with the Weierstrass function and its derivative. **One of the purposes of the present paper will be to demonstrate how this parametrization can be extended to all the variables dX^{i} in the cubic algebraic equation.** It will be demonstrated in section 3 how an embedded system of cubic algebraic equations can be obtained, and each equation from this sequence will be for the algebraic subvariety of the solutions - i.e. the first equation will be for the algebraic variety of n variables, the second one - for $(n-1)$ variables,...., the last one will be only for the dX^{1} variable**. Thus a qualitatively new moment appears in the investigation of elliptic curves and cubic algebraic equations - it turns out that it is possible to parametrize not only a plane cubic algebraic curve $y^{2}=4x^{3}-g_{2}x-g_{3}$, but also a multi - variable cubic algebraic equation.** In section 4 it has been proved that yet the situation is different from the standard case - the obtained "uniformization" functions for the differentials dX^{1}, dX^{2}, dX^{3} are complicated irrational functions of the Weierstrass function $\rho(z)$, its derivative $\rho^{'}(z)$, the (covariant) metric tensor and affine connection components, which depend on the global coordinates X^{1}, X^{2}, X^{3}. Therefore, it might seem that due to the dependence on the global coordinates one cannot assert that the functions dX^{1}, dX^{2}, dX^{3} are "uniformization functions", since they should depend **only** on the complex coordinate z. But in section 5 it has been proved that the obtained expressions can be transformed to a system of first - order nonlinear differential equations in respect to X^{1}, X^{2}, X^{3} and since a solution of this system always exists, one may write down $X^{1}=X^{1}(z)$, $X^{2}=X^{2}(z)$, $X^{3}=X^{3}(z)$. This solves completely the problem about the existence of "uniformization functions". On the other side, the generalized coordinates depend also on the s. c. "initially given" coordinates $\mathbf{x}=(x_{1}$, x_{2}, $x_{3})$, so the complex function dependence is more complicated, i. e. $X^{1}=X^{1}(\mathbf{x}(z))$, $X^{2}=X^{2}(\mathbf{x}(z))$, $X^{3}=X^{3}(\mathbf{x}(z))$. In section 6 it was shown, introducing special notations and the s.c. "one - dimensional Poisson bracket", that additionally a system of first - order nonlinear differential equations may be written also in respect to the initial coordinates x_{1}, x_{2}, x_{3}. Sections 7, 8 and 9 treat an important issue, which is purely a mathematical one, but can have important physical applications - how the complex coordinate dependence of the initial and general coordinates can be extended from one to

two complex coordinates z, v. In section 7 it was shown that this cannot be done by assuming that $X^i = X^i(x(z), z, v)$, since then the obtained system of equations will be conradictable. Intuitively, this conclusion can be justified because it is unnatural for the complex coordinate z to be in a "privilleged" position because of the dependence of the initial coordinates only on z.Therefore, it is much more natural if the complex coordinate dependence is realized in the following way: $X^i = X^i(x(z, v), z)$. By performing a full analysis of the system of equations, in Sections 8 and 9 it was shown that this time no contradiction and incompatibility in the system of equations appear.

Throughout all the sections from 3 to 9, the approach was applied by assuming that the second differentials of the generalized coordinates are zero $d^2X^i = 0$. This may seem to be a serious restriction, but **it was necessary to be imposed in order to be able to construct the algorithm for finding the solutions of the cubic equation**. So it is not realistic to hope that just at once such an algorithm may be constructed, without making any assumptions whatsoever. The important point is another - since the main tool of the proposed approach is the linear - fractional transformation, will this approach be applicable also for the more complicated cases of 1. again the cubic algebraic equation of reparametrization invariance, but with $d^2X^i \neq 0$ and 2. the system of **cubic and quartic Einstein's algebraic equations**. In the subsequent second and third parts of this paper, it will be shown that the approach continues to work, although with some modifications and unfortunately, with considerable (but not unsurmountable) technical difficulties. In these two cases, the application of this approach will require: 1. **enlarging the algebraic variety** of solutions with the addition of the second differentials (first case) or the first and the second derivatives (the second case) 2. Applying a **multi - component linear - fractional transformation**, which accounts for all the variables in the extended algebraic variety. This will result in several coefficient functions in front of the highest degrees, all of which have to be set up to zero. In other words, if in the presently investigated case we will have **only one** cubic equation with an algebraic variety of n, $(n-1)$, ..., 2, 1 dimensions after **each** application of the linear - fractional transformation, then after each case one would have **at least** two algebraic equations, one of which may be of third, and the other - of second order. 3. Since the originally given multi - variable algebraic equation is transformed by means of a linear - fractional transformation in respect to all the variables in the algebraic variety, which includes also first derivatives (first case) and first and second derivatives (second case), **additionally,** if one sets up $d^2X^i = Z^i$ and $dX^i = Y^i$, one has to take into account also that $dY^i = Z^i$, which further will be called the "compatibility system of equations". This would result in a system of differential equations in respect to the coefficient functions in the multi - component linear - fractional transformation. Even for the simpler (first) case with $d^2X^i \neq 0$, the solution of this system will be connected with considerable difficulties. However, yet it will turn out for the first case that such a (unique!) solution can found, and it helps to express **uniquely** the coefficient functions in the linear - fractional transformation for d^2X^i through the ones in the transformation for dX^i.

There are a number of physical applications of the algebraic geometry formalism, concerning **gravitational theories with extra dimensions**, which will be considered

in the last section 10 of this paper. These applications and most of the corresponding equations are not directly related to parametrization with the Weierstrass function, but the approach in principle follows the same main idea of introducing a more general contravariant tensor. In our opinion, one of the most important applications is related to the so called "exponential damping" of the mass m_0 of the visible three - brane in a fundamental higher - dimensional brane theory, which has been considered in the pioneering paper of L. Randall and R. Sundrum and which results in an "exponentially damped" physical mass $m = m_0 e^{-kr_{-}\pi}$. If one assumes $e^{kr_{-}\pi} \simeq 10^{15}$, then this in fact turns out to be the mechanism for producing physicall mass scales not far from the Planck scale $\sim 10^{19}$ GeV. The problem however is that the contravariant metric components play the role of a "coupling" parameter in the effective action of the visible brane. So it has been shown in this paper, that for the same scalar (constant and negative) gravitational curvature of a five - dimensional space with an embedded flat $4D$ Minkowski space, other contravariant components $\widetilde{g}^{ij}$ may be found, which are proportional to the components g^{ij}, i.e. $\widetilde{g}^{ij} = l(\mathbf{x})\widetilde{g}^{ij}$ $(\mathbf{x} \equiv \overrightarrow{\mathbf{x}} = (x_1, x_2, x_3, x_4))$ and the function of proportionality $l(\mathbf{x})$ (it will be called a **length function**) may be found as an **(exact analytical) solution of a quasilinear differential equation in partial derivatives.** Thus, it may be supposed that the physical Higgs mechanism for dynamical mass generation may turn out to be more complicated, and so in principle, **there might be a variety of generated physical masses.** At least, this should strictly follow from the whole mathematical construction and mostly from the fact that solution of a differential equation of partial derivatives may be **any function,** depending on the first integrals of the characteristic system of equations. It is perhaps from a physical point of view interesting to note that in one of found solutions for l (form. (10.197))

$$l^2 = \frac{1}{1 - const.\ e^{24\ k\ \varepsilon\ y}} \qquad \varepsilon = \pm 1 \quad , \tag{1.13}$$

the "scale function" will indeed be equal to one (i.e. we have the usual gravitational theory with $\widetilde{g}^{ij} = g^{ij}$ and $l = 1$) for $\varepsilon = -1$ and $y \to \infty$ (the s.c. infinite extra - dimensions), but for $\varepsilon = +1$ there will be even a decrease of the "length function" due to the exponenrtial factor in the denominator. Yet, the derivation of a justifiable physical result from a solution of the characteristic equation poses one more interesting problem, this time **in reference to the invariance of the low - energy type I string theory effective action under both compactification from a ten -dimensional spacetime to five - dimensional one and also (again) rescaling of the contravariant metric components,** which has been investigated also in section 10. The peculiar moment here is that together with the compactification (which is the standard known approach so far), here the rescaling is also taken into consideration, which again leads to the construction of **quasilinear differential equations in partial derivatives in respect to the function** $l(x)$. Here the important fact is that these equations are non - contradictable in the limit $l = 1$, which means that the usual and known relations between the parameters in the low - energy type I string action in this limit are obtained. However, from the solutions of the characteristic equations the limit $l = 1$ can be set up and can give some

new relations between the parameters. Thus, if any information about the string scale m_s, the string coupling constant λ and the electromagnetic coupling constant g_5 is available, then in principle the obtained solutions should result also in the unit lenght scale $l = 1$, if the known theory of gravity is correct. If this does not happen, then under certain energies, string scales, electromagnetic and string couplings there might be deviations from the case $l = 1$. It turns out, the standard gravitational theory may be tested, since it is related to the above mentioned parameters.

2 BASIC KNOWLEDGE ABOUT ALGEBRAIC EQUATIONS IN GRAVITY THEORY

2.1. THE INTRODUCED CONNECTION $\widetilde{\Gamma}_{kl}^{s} \equiv \frac{1}{2} dX^i dX^s (g_{ik,l} + g_{il,k} - g_{kl,i})$

This section has the purpose to review the possible application of algebraic geometry and theory of algebraic equations in gravity theory. A part of the exposition will be based on the paper [10] (see also [23]), but some new generalizations will be presented.

It is known in gravity theory that the knowledge of the **metric tensor** g_{ij} determines the space - time geometry, which means that the **Christoffell connection**

$$\Gamma_{ik}^{l} \equiv \frac{1}{2} g^{ls} (g_{ks,i} + g_{is,k} - g_{ik,s}) \tag{2.1}$$

and the **Ricci tensor**

$$R_{ik} = \frac{\partial \Gamma_{ik}^{l}}{\partial x^l} - \frac{\partial \Gamma_{il}^{l}}{\partial x^k} + \Gamma_{ik}^{l} \Gamma_{lm}^{m} - \Gamma_{il}^{m} \Gamma_{km}^{l} \tag{2.2}$$

can be calculated. **Now suppose that one knows the the Ricci tensor** $R_{ij} = R_{ij}(x)$ **as a function of the space - time coordinates, but one would like to find the Christoffell connection** Γ_{ij}^{k}. Then one can consider the defining equation (2.2) in two different ways

1. As a **first - order system of nonlinear differential equations in partial derivatives**

$$\frac{\partial \Gamma_{ij}^{k}}{\partial x^l} = F(R_{ik}, \Gamma_{il}^{m} \Gamma_{km}^{l}) \tag{2.3}$$

with a quadratic nonlinearity in the components of the Christoffell connection.The number of equations is equal to the number of components of the symmetric tensor R_{ik}, which for an n-dimensional space - time is $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ and the number of the unknown variables is equal to the number of the Christoffell connection components, which is $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right] n$.

Note that even if R_{ik} have been calculated from the initially given connection Γ^k_{ij}, the solution of the system of differential equations (2.3) will not give only the initial connection, but also annother connections. The reason is that the corresponding boundary conditions for the system (2.3) have not been fixed, thus there is no theorem for uniqueness of the solution.

2. As a system of $\left(\begin{array}{c} n \\ 2 \end{array} \right) + n$ algebraic equations for an algebraic variety (i.e. the variety of the unknown variables in the algebraic equation) of $\left[\left(\begin{array}{c} n \\ 2 \end{array} \right) + n \right] n$ variables (the components of the Christoffell connection) plus $\left[\left(\begin{array}{c} n \\ 2 \end{array} \right) + n \right] n^2$ derivatives $\Gamma^k_{ij,l}$. Thus the total number of the elements of the algebraic variety is $\frac{n^2(n+1)^2}{2}$. Note that the derivatives in an algebraic equation can also enter as elements of the algebraic variety.

One might therefore expect that there will be a whole variety of connections, which will give the same Ricci tensor.

It is useful to remember also from standard textbooks on gravity theory [24] that the s. c. **Christoffel connection of the first kind:**

$$\Gamma_{i;kl} \equiv g_{im}\Gamma^m_{kl} = \frac{1}{2}(g_{ik,l} + g_{il,k} - g_{kl,i}) \tag{2.4}$$

is obtained from the expression for the zero covariant derivative $0 = \nabla_l g_{ik} = g_{ik,l} - g_{m(i}\Gamma^m_{k)l}$ **without assuming** that $g_{ij}g^{jk} = \delta^k_i$. **After the derivation** of this formulae it becomes clear that the consistency of the definition (2.1) of the Christoffell connection (derivable by contracting (2.4) with the contravariant metric tensor g^{is}) would require the existence of an inverse contravariant metric tensor. But then by contraction of (2.4) with **another** conravariant tensor field $\widetilde{g}^{is}$ one might as well define **another connection:**

$$\widetilde{\Gamma}^s_{kl} \equiv \widetilde{g}^{is}\Gamma_{i;kl} = \widetilde{g}^{is}g_{im}\Gamma^m_{kl} = \frac{1}{2}\widetilde{g}^{is}(g_{ik,l} + g_{il,k} - g_{kl,i}) \quad , \tag{2.5}$$

not consistent with the initial metric g_{ij}. Clearly the connection (2.5) is defined under the assumption that the metric tensor g_{ij} does not have on inverse one and therefore $\widetilde{g}^{is}g_{im} \equiv l^s_m$. The use of such a convention may seem strange, but it was shown above that it is fully consistent with the defining equations and as will be demonstrated further, this more general definition will allow some important generalization and possibilities for obtaining more general solutions even of the Einstein's equations.

In fact, the definition $\widetilde{g}^{is}g_{im} \equiv l^s_m$ turns out to be inherent to gravitational physics. For example, in the **projective formalism** one decomposes the standardly defined metric tensor (with $g_{ij}g^{jk} = \delta^k_i$) as

$$g_{ij} = p_{ij} + h_{ij} \quad , \tag{2.6}$$

together with the additional assumption that the two subspaces, on which the **projective tensor** p_{ij} and the tensor h_{ij} are defined, are orthogonal. This means that

$$p_{ij}h^{jk} = 0 \quad . \tag{2.7}$$

As a consequence

$$p_{ij}p^{jk} = \delta_i^k - \frac{1}{e}h_{ij}h^{jk} \neq \delta_i^k \tag{2.8}$$

and **under this condition** the projective connection $\overline{\overline{\Gamma}}_{ik}^{l}$ is defined in a similar way

$$\overline{\overline{\Gamma}}_{ik}^{l} \equiv \frac{1}{2}p^{ls}(p_{ks,i} + p_{is,k} - p_{ik,s}) \quad . \tag{2.9}$$

Another example of gravitational theories with more than one connection are the so called **theories with affine connections and metrics** [13], in which there is one connection $\Gamma_{\alpha\beta}^{\gamma}$ for the case of a parallel transport of covariant basic vectors $\nabla_{e_\beta}e_\alpha = \Gamma_{\alpha\beta}^{\gamma}$ e_γ and a **separate connection** $P_{\alpha\beta}^{\gamma}$ for the contravariant basic vector e^γ, the defining equation for which is $\nabla_{e_\beta}e^\alpha = P_{\gamma\beta}^{\alpha}\ e^\gamma$. Since $e_\alpha e^\beta \equiv f_\alpha^\beta(x)$ for such theories, one can obtain after covariant differentiation that the two connections are related in the following way [13]

$$f_{j,k}^i = \Gamma_{jk}^l\ f_l^i + P_{lk}^i f_{jk}^l \qquad ; \qquad (f_{j,k}^i = \partial_k f_j^i) \quad . \tag{2.10}$$

However, the function $f_j^i(x)$ cannot be determined from any physical considerations, but similarly to the proposed here approach one can write down the gravitational Lagrangian in terms of the two connections and make it equal to the gravitational Lagrangian with only one connection. Then the defining relation can again be expected to be an algebraic equation (at least quadratic) in respect to the two connections $\Gamma_{\alpha\beta}^{\gamma}$ and $P_{\alpha\beta}^{\gamma}$.

In the present case the introduced connection (2.5) should not be identified with the connection $P_{\alpha\beta}^{\gamma}$, since the connection $\widetilde{\Gamma}_{kl}^{s} \equiv \widetilde{g}^{is}\Gamma_{i;kl}$ is introduced by means of modifying the contravariant tensor and not on the base of any separate parallel transport. Moreover, the freedom in determination of $\widetilde{\Gamma}_{kl}^{s}$ would contradict the relation between the two connections (2.10) in case if such an identification is made.

In [10] a definite choice of the contravariant tensor $\widetilde{g}^{ij}$ was made in the form of the factorized product $\widetilde{g}^{ij} \equiv dX^i dX^j$, where $X^i = X^i(x_1, x_2, ..., x_n)$ are some generalized coordinates, which can be regarded as an $n-$ dimensional hypersurface, defining a transition from an **initially** defined set of coordinates $x_1, x_2, ..., x_n$ on the chosen manifold to **another set of the (generalized) coordinates** X^i ($i = 1, 2.....n$; n - the dimension of spacetime). If dX^i are assumed to be vectors, lying in the tangent space to the hypersurface, the defined earlier connection (2.5) assumes the form

$$\widetilde{\Gamma}_{kl}^{s} \equiv dX^i dX^s \Gamma_{i;kl} = \frac{1}{2}dX^i dX^s(g_{ik,l} + g_{il,k} - g_{kl,i}) \quad . \tag{2.11}$$

The connection $\widetilde{\Gamma}_{kl}^{s}$ has two important properties, already mentioned in the Introduction and proved in Appendix A: it has an affine transformation law under change of variables and moreoever, it is an equiaffine connection [15, 16], which means that it can be represented as a gradient of a scalar quantity, i.e. $\widetilde{\Gamma}_{ks}^{s} = \partial_k lge$, where e is some scalar quantity. Since by definition a connection is an equiaffine if and only if the corresponding Ricci tensor $\widetilde{R}_{ij}$ is a symmetric one in respect to the indices i and j, the usual formulae for the

Ricci tensor can be used, but with the newly defined connection $\widetilde{\Gamma}_{kl}^{s}$ instead of the usual Christoffell connection Γ_{kl}^{s}.

In the formal mathematical sense, (2.11) can be treated as a system of quadratic algebraic equations in respect to the differentials dX^{i} and under known connection components $\widetilde{\Gamma}_{kl}^{s}$. Such an interpretation would be incorrect because for fixed indices k and l and a varying indice s (2.11) represents also a system of linear algebraic equations in respect to the variables $G_{i} \equiv g_{ik,l} + g_{il,k} - g_{kl,i}$ (k, l - fixed). **Since the determinant** $det \parallel \widetilde{g}^{ij} \parallel = det \parallel dX^{i}dX^{j} \parallel \equiv 0$**, for non - zero** $\widetilde{\Gamma}_{kl}^{s}$ **there will be no solutions of this system of equations in respect to** G_{i}**. This means that the components of the connection** $\widetilde{\Gamma}_{kl}^{s}$ **cannot be fixed arbitrarily,** but instead one can choose the differential functions dX^{i} and the derivatives of the metric tensor $g_{ij,k}$. There is one exception - for certain k and l one can choose all the connection components $\widetilde{\Gamma}_{kl}^{s}$ for all $s = 1, 2......n$ to be zero. Then the system of equations (2.11) for $\widetilde{\Gamma}_{kl}^{s} = 0$ will be satisfied for **arbitrary** $g_{ij,k}$ with the given k and l.

2.2. BASIC ALGEBRAIC EQUATIONS IN GRAVITY THEORY

Now if one applies again the new definition $\widetilde{g}^{ij} \equiv dX^{i}dX^{j}$ of the contravariant tensor in respect to the Ricci tensor, then the following **fourth - degree algebraic equation** can be obtained

$$R_{ik} = dX^{l}\left[g_{is,l}\frac{\partial(dX^{s})}{\partial x^{k}} - \frac{1}{2}pg_{ik,l} + \frac{1}{2}g_{il,s}\frac{\partial(dX^{s})}{\partial x^{k}}\right] +$$

$$+\frac{1}{2}dX^{l}dX^{m}dX^{r}dX^{s}\left[g_{m[k,t}g_{l]r,i} + g_{i[l,t}g_{mr,k]} + 2g_{t[k,i}g_{mr,l]}\right] \qquad , \qquad (2.12)$$

where p is the scalar quantity

$$p \equiv div(dX) \equiv \frac{\partial(dX^{l})}{\partial x^{l}}, \qquad (2.13)$$

which "measures" the **divergency** of the vector field dX. The algebraic variety of the equation consists of the differentials dX^{i} and their derivatives $\frac{\partial(dX^{s})}{\partial x^{k}}$.

In the same spirit one can investigate the problem whether the gravitational Lagrangian in terms of the new contravariant tensor can be equal to the standard representation of the gravitational Lagrangian. The condition for the equivalence of the two representations gives a **cubic algebraic equation** in respect to the algebraic variety of the first differential dX^{i} and the second one $d^{2}X^{i}$ [10]

$$dX^{i}dX^{l}\left(p\Gamma_{il}^{r}g_{kr}dX^{k} - \Gamma_{ik}^{r}g_{lr}d^{2}X^{k} - \Gamma_{l(i}^{r}g_{k)r}d^{2}X^{k}\right) - dX^{i}dX^{l}R_{il} = 0 \qquad . \qquad (2.14)$$

From this equation the following important generalization can be made: Let us assume that the components Γ_{ik}^{l} of the standard Christoffell connection in the above equation are not given, but instead only the metric tensor g_{ij}, its partial derivatives and the scalar

curvature R are given. Then the above written equation assumes the form of a **fifth - degree algebraic equation**

$$dX^i dX^l dX^r dX^s \left(p\Gamma_{s;il} g_{kr} dX^k - \Gamma_{s;ik} g_{lr} d^2X^k - \Gamma_{s;l(i} g_{k)r} d^2X^k \right) - R = 0 \quad . \tag{2.15}$$

The mathematical treatment of such equations is known since the time of Felix Klein's famous monograph [25], published in 1884. A way for resolution of such equations on the base of earlier developed approaches in by means of reducing the fifth - degree equations to the so called **modular equation** has been presented in the more recent monograph of Prasolov and Solov'yev [9]. It should be stressed that the presented in [10] (standardly known) method for parametrization with the Weierstrass function of a cubic algebraic equation should be considered as **just one possibility** for solution of third - order algebraic equations. Some other methods for solution of third-, fifth- and higher- order algebraic equations, based on resolvents with a group of linear substitutions [26, 27] date from the end of the previous century (see particularly [28] and also [29] for a complete and detailed description of the properties of **elliptic, theta and modular functions** from the viewpoint of the earliest developments more that 100 years ago). It is known as well that solutions of $n-$ th degree algebraic equations can be presented also in **theta - constants** [30] and also in **special functions** [31]. The first possibilitty, related also to modular equations, is interesting in view of the not yet proven hypothesis in the paper by Kraniotis and Whitehouse [8] that **"all nonlinear solutions of general relativity are expresed in terms of theta - functions, associated with Riemann - surfaces"**.

In [10] the Einstein's equations in vacuum **for the general case** were presented under the assumption about the newly defined contravariant metric tensor

$$0 = \widetilde{R}_{ij} - \frac{1}{2} g_{ij} \widetilde{R} = \widetilde{R}_{ij} - \frac{1}{2} g_{ij} dX^m dX^n \widetilde{R}_{mn} =$$

$$= -\frac{1}{2} p g_{ij} \Gamma^r_{mn} g_{kr} dX^k dX^m dX^n + \frac{1}{2} g_{ij} (\Gamma^r_{km} g_{nr} + \Gamma^r_{n(m} g_{k)r}) d^2X^k dX^m dX^n +$$

$$+ p\Gamma^r_{ij} g_{kr} dX^k - (\Gamma^r_{ik} g_{jr} + \Gamma^r_{j(i} g_{k)r}) d^2X^k \quad . \tag{2.16}$$

This equation is again a system of **cubic equations** or it can analogously to (2.15) be presented as a system of **fifth - degree** algebraic equations. **In addition, if the differentials dX^i and d^2X^i are known, but not the covariant tensor g_{ij}, the same equation can be considered also as a cubic algebraic equation in respect to the algebraic variety of the metric tensor components g_{ij} and their first derivatives $g_{ij,k}$.**

2.3. SOME GENERAL ANALYSIS OF THE N - DIMENSIONAL SYSTEM OF EQUATIONS $g_{ij}\widetilde{g}^{jk} = \delta^k_i$

Of course, a more general theory with the definition of the contravariant tensor as $\widetilde{g}^{ij} \equiv dX^i dX^j$ should contain in itself the standard gravitational theory with $g_{ij} g^{jk} = \delta^k_i$. From

a mathematical point of view, this should be performed by **considering the intersection of the cubic algebraic equation (2.16) (or of its fifth - degree counterpart) with the system of** n^2 **quadratic algebraic equations for the algebraic variety of the** n variables

$$g_{ij}dx^j dx^k = \delta_i^k \quad . \tag{2.17}$$

It might seem that the system of equations (2.17) does not have solutions in respect to g_{ij} (and thus no solutions of the Einstein's equations can be found for the standard case), since the determinant $det \parallel dx^i dx^j \parallel_{i,j=1,..n}= 0$ equals to zero! Such a statement would be however **wrong**, since (2.17) should not be considered as a system of linear algebraic equations (for which the unknown variables are contained in a **vector - column and therefore all the theorems of linear algebra can be applied),** but rather than that as an **operator equation of the kind** $Y_{ij}g^{jk} = \delta_i^k$ **with unknown variables** $Y_{ij} \equiv g_{ij}$, **which constitute a matrix and not a vector - column! Therefore the linear algebra theorems cannot be applied.** A general theory of operator equations of the type $AX = XB$ (X- the unknown matrix) is developed in the known monograph of Gantmaher [20]. In Appendix C it will be shown how the **operator system of equations** $Y_{ij}g^{jk} = \delta_i^k$ can be transformed to a system $\widetilde{A}_{ij}\widetilde{Y}^{\ j} = \delta_i$ (δ_i is a **vector - column**, consisting only of 1 and 0; $i, j = 1, 2..., \frac{n(n+1)}{2}$). The method at first will be demonsrated for the simple case when $i, j = 1, 2, 3$; subsequently the method will be generalized to arbitrary $n-$dimensions, i.e. it will be explicitely shown how the matrix $\widetilde{A}_{ij}$ can be constructed for the general $n-$dimensional case. The advantage of the proposed approach is that it will allow explicitely to answer the question: will the operator system of equations $Y_{ij}g^{jk} = \delta_i^k$ have a solution in the case, when the contravariant metric tensor $\widetilde{g}^{jk}$ is chosen in the form of a factorized product $\widetilde{g}^{jk} \equiv dX^j dX^k$? It will become evident that this choice, inspite of the zero determinant $\parallel dx^i dx^j \parallel_{i,j=1,..n}$, **does not** put a restriction on the solvability of the considered **predetermined** system of n^2 equations (for different values of the indices i and k) for the $\begin{pmatrix} n \\ 2 \end{pmatrix} + n = \frac{n(n+1)}{2}$ variables g_{ij}.

Now let us consider some more general properties when of the system $g_{ij}\widetilde{g}^{jk} = \delta_i^k$ in the general $n-$dimensional case. Since the number of equations is greater than the number of variables, from the n^2 equations one can select $n^2 - n$ equations with different values of i and k, for which the R. H. S. will be zero. Yet the number of the chosen equations remains to be greater than the number of variables $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$, which is confirmed by the equality

$$(n^2 - n) - \left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right] = \frac{n(n-3)}{2} \quad , \tag{2.18}$$

fulfilled for $n > 3$. **Only** for the case $n = 3$, as it will be shown in Appendix C, the number of the equations $n^2 - n = 6$ with a zero R. H. S. becomes exactly equal to the number of variables $\begin{pmatrix} n \\ 2 \end{pmatrix} + n = \frac{n(n+1)}{2} = \frac{3.\ 4}{2} = 6$. Therefore for the case of an arbitrary

n from these $n^2 - n$ equations one can choose $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n\right]$ equations. In each row of the $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n\right] \times \left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n\right]$ matrix of coefficient functions of this system there will be only n functions $\widetilde{g}^{jk}$, the rest $\begin{pmatrix} n \\ 2 \end{pmatrix}$ of the elements will be zero. The determinant of the matrix will be equal to zero, a proof of which in the general case of an arbitrary n (and also for $n = 3$) is presented in Appendix C. Therefore, **the solutions** g_{ij} of this $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ dimensional homogeneous system of equations with a zero determinant are **arbitrary.**

Now we are left with $\frac{n(n-3)}{2}$ equations (again with $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ variables) with a zero R. H. S. (we shall call it the **first system of equations**) and with another n equations (the **second system**) with a R. H. S. of each equation, equal to 1. Since the number of variables $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ in the **first system** is greater than the number $\frac{n(n-3)}{2}$ of equations and

$$\begin{pmatrix} n \\ 2 \end{pmatrix} + n - \frac{n(n-3)}{2} = 2n \quad , \tag{2.19}$$

again one can treat as unknown only $\frac{n(n-3)}{2}$ variables and transfer the rest $2n$ variables in the R. H. S., which will become different from zero. Analogously, for the **second system** one can treat as unknown only n variables and transfer the rest $\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2}$ variables in the R. H. S. If this R. H. S. is different from zero and moreover, the determinant of the coefficient functions $\widetilde{g}^{ij}$ is also different from zero, **then one can find unique solutions for these n variables** g_{ij}. Now comes the most important point of the proof: Since

$$n + \frac{n(n-1)}{2} > 2n \quad , \tag{2.20}$$

one can take all n of the uniquely found solutions plus n more unfixed (freely varied) variables from the R. H.S. of the second system and "plunge" them into the R. H. S. of the first system. Let us remember that the first system has $\frac{n(n-3)}{2}$ unknown variables, but one may note that $\begin{pmatrix} n \\ 2 \end{pmatrix} + n - 2n = \frac{n(n-3)}{2}$ variables from the second system have not been transfered in the R. H. S. of the first system. Therefore, one can choose these $\frac{n(n-3)}{2}$ variables to be the unknown variables for the first system, and if the determinant of the coefficient functions is nonzero, an unique solution can be found for them.

As a whole , one would have a maximum of $n+ \frac{n(n-3)}{2} = \frac{n(n-1)}{2}$ uniquely fixed variables and the rest of the variables may be varied freely.

2.4. ALGEBRAIC EQUATIONS FOR A GENERAL CON-

TRAVARIANT METRIC TENSOR

Let us write down the algebraic equations for all admissable parametrizations of the gravitational Lagrangian for the general case of contravariant tensor $\widetilde{g}^{ij}$:

$$\widetilde{g}^{i[k}\widetilde{g}^{l]s}\Gamma_{ik}^{r}g_{rs} + \widetilde{g}^{i[k}\widetilde{g}^{l]s}\left(\Gamma_{ik}^{r}g_{rs}\right)_{,l} +$$

$$\widetilde{g}^{ik}\widetilde{g}^{ls}\widetilde{g}^{mr}g_{pr}g_{qs}\left(\Gamma_{ik}^{q}\Gamma_{lm}^{p} - \Gamma_{il}^{p}\Gamma_{km}^{q}\right) - R = 0 \quad . \tag{2.21}$$

This equation is again a **cubic algebraic equation** in respect to the algebraic variety of the variables $\widetilde{g}^{ij}$ and $\widetilde{g}_{,k}^{ij}$, and the number of variables in the present case is much greater than in the previous case for the contravariant tensor $\widetilde{g}^{ij} \equiv dX^{i}dX^{j}$. If the connection is assumed to be the "tilda connection" (2.11) $\widetilde{\Gamma}_{kl}^{s} \equiv dX^{i}dX^{s}\Gamma_{i;kl}$, then the same equation can be regarded as a **fifth - degree equation** in respect to the contravariant tensor $\widetilde{g}^{ij}$ and its derivatives $\widetilde{g}_{,k}^{ij}$ and at the same time as a **fourth - degree algebraic equation** in respect to the covariant metric tensor g_{ij} and its first and second partial derivatives.

$$\widetilde{g}^{i[k}\widetilde{g}^{l]s}\widetilde{g}^{rt}\Gamma_{t;ik}\ g_{rs} + \widetilde{g}^{i[k}\widetilde{g}^{l]s}\widetilde{g}_{,l}^{rt}\left(\Gamma_{t;\ ik}g_{rs}\right) +$$

$$+\widetilde{g}^{i[k}\widetilde{g}^{l]s}\widetilde{g}^{rt}g_{rs}\left(\Gamma_{t;\ ik}\right)_{,l} + \widetilde{g}^{i[k}\widetilde{g}^{l]s}\widetilde{g}^{rt}\Gamma_{t;ik}\ g_{rs,l} +$$

$$+\widetilde{g}^{ik}\widetilde{g}^{ls}\widetilde{g}^{mr}\widetilde{g}^{qt_1}\widetilde{g}^{pt_2}g_{pr}g_{qs}\left[\Gamma_{t_1;ik}\Gamma_{t_2;lm} - \Gamma_{t_1;mk}\Gamma_{t_2;il}\right] - R = 0 \quad . \tag{2.22}$$

Note that the consideration of this equation either as a **fourth-** or as a **fifth-** degree equation means that the algebraic equation will be with **coefficient functions** and not with **number coefficients**, as are the equations in the standardly known **arithmetic theory**. This difficulty can be overcomed if the above equation is treated **simultaneously in respect to the algebraic variety of the covariant and the contravariant variables** - then (2.22) is an algebraic equation with number coefficients . Of course, the algebraic treatment of an equation of **nineth degree** is not quite known.

Similarly the Einstein's equations (2.16) can be written as a system of **fifth - degree algebraic equations in respect to the contravariant variables** and as **fifth - degree degree equations in respect to the covariant variables (which is the difference from the previous case). Note that in such a representation for the general case, when already the determinant of the components of the contravariant metric is non - degenerate, i. e.** $det \parallel \widetilde{g}^{jk} \parallel \neq 0$, **the additional equation** $g_{ij}\widetilde{g}^{jk} = \delta_{i}^{k}$ **can also be considered together with the Einstein's equations. From an algebro - geometric point of view, this turns out to be a problem about the intersection of the Einstein's algebraic equations with the system of** n^{2} **(linear) hypersurfaces for the** $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n\right]$ **variables of the covariant tensor and** $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n\right]$ **variables for the contravariant ones.** If the corresponding equations are cubic ones, this is an analogue to the well - known problem in algebraic geometry about the intersection of a cubic surface with a straight line. In this aspect one can point out the following three important problems:

1. One can find solutions of the system of Einstein's equations not as solutions of a system of nonlinear differential equations, but as **elements of an algebraic variety, satisfying the Einstein's algebraic equations.** The important new moment is that **this gives an opportunity to find solutions of the Einstein's equations both for the components of the covariant metric tensor g_{ij} and for the contravariant ones $\widetilde{g}^{jk}$. This means that solutions may exist for which $g_{ij}\widetilde{g}^{jk} \neq \delta_i^k$.** In other words, a classification of the solutions of the Einstein's equations can be performed in an entirely new and nontrivial manner:

a) **Under a given contravariant tensor, the covariant tensor and its derivatives have to be found from the algebraic equation**.

b) **Under a given covariant tensor, the contravariant tensor and its derivatives have to be found from the corresponding algebraic equation**.

c) **If both the covariant and the contravariant tensors are considered to be unknown, then their components have to be found from the corresponding algebraic equation with number coefficients. This is what is really meant by "solutions of the Einstein's equations both for the components of the covariant metric tensor g_{ij} and for the contravariant ones $\widetilde{g}^{jk}$."**

2. The standardly known solutions of the Einstein's equations can be obtained as an intersection variety of the Einstein's algebraic equations with the system of linear hypersurfaces $g_{ij}\widetilde{g}^{jk} = \delta_i^k$. However, the strict mathematical proof that such an intersection of the system of tenth - degree equations (a fifth - degree for the variety of the covariant variables and a fifth - degree for the contravariant ones) with the system of linear hypersurfaces will give the standard case is **still lacking.**

3. The condition for the zero - covariant derivative of the covariant metric tensor $\nabla_k g_{ij} = 0$ and of the contravariant metric tensor $\nabla_k \widetilde{g}^{ij} = 0$ can be written in the form of the following **cubic algebraic equations** in respect to the variables g_{ij}, $g_{ij,k}$ and $\widetilde{g}^{ls}$ **:**

$$\nabla_k g_{ij} \equiv g_{ij,k} - \widetilde{\Gamma}^l_{k(i}g_{j)l} = g_{ij,k} - \widetilde{g}^{ls}\Gamma_{s;k(i}g_{j)l} = 0 \tag{2.23}$$

and

$$0 = \nabla_k \widetilde{g}^{ij} = \widetilde{g}^{ij}_{,k} + \widetilde{g}^{r(i}\widetilde{g}^{j)s}\Gamma_{r;sk} \quad . \tag{2.24}$$

The first equation (2.23) is linear in respect to $\widetilde{g}^{ls}$ and quadratic in respect to g_{ij}, $g_{ij,k}$, while the second equation (2.24) is linear in respect to g_{ij}, $g_{ij,k}$ and quadratic in respect to $\widetilde{g}^{ls}$- therefore both equations are cubic in respect to these variables.

Since the treatment of the above cubic algebraic equations is based on singling out one variable, let us rewrite equation (2.22) for the effective parametrization of the gravitational action for the case of diagonal metrics $g_{\beta\beta}$ and $\widetilde{g}^{\alpha\alpha}$, singling out the variable $\widetilde{g}^{44}$:

$$A(\widetilde{g}^{44})^3 + B_\alpha(\widetilde{g}^{44})^2\widetilde{g}^{\alpha\alpha} + C_{\alpha\alpha}\widetilde{g}^{44}\widetilde{g}^{\alpha\alpha} + (\Gamma^\alpha_{44}g_{\alpha\alpha})\widetilde{g}^{44}\widetilde{g}^{\alpha\alpha}_{,\alpha} +$$

$$+ D_{\alpha\gamma}\widetilde{g}^{44}\widetilde{g}^{\alpha\alpha}\widetilde{g}^{\gamma\gamma} + F_{\alpha\gamma} = 0 \quad , \tag{2.25}$$

where the coefficient functions A, B_α, $C_{\alpha\alpha}$, $D_{\alpha\gamma}$ and the free term $F_{\alpha\gamma}$ denote the following expressions:

$$A \equiv g_{4[4}g_{p]p}\Gamma^4_{44}\Gamma^p_{44} \quad , \tag{2.26}$$

$$B_\alpha \equiv g_{4[4}g_{\alpha]\alpha}\Gamma^4_{44}\Gamma^\gamma_{4\gamma} + g_{\alpha\alpha}g_{44}(\Gamma^4_{44}\Gamma^\alpha_{\alpha 4} - \Gamma^4_{4\alpha}\Gamma^\alpha_{44}) +$$

$$+g_{\alpha\alpha}g_{44}\Gamma^\alpha_{44}\Gamma^4_{\alpha 4} + g_{44}g_{pp}\Gamma^p_{\alpha\alpha}\Gamma^4_{44} \quad , \tag{2.27}$$

$$C_{\alpha\alpha} \equiv (\Gamma^\alpha_{44}g_{\alpha\alpha})_{,\alpha} + \left(\Gamma^4_{\alpha\alpha}g_{44}\right)_{,4} \quad , \tag{2.28}$$

$$D_{\alpha\gamma} \equiv g_{\alpha\alpha}g_{\gamma\gamma}(\Gamma^\gamma_{44}\Gamma^\alpha_{\alpha\gamma} - \Gamma^\gamma_{4\alpha}\Gamma^\alpha_{4\gamma}) + g_{44}g_{\gamma\gamma}\left[2\Gamma^4_{\alpha\alpha}\Gamma^\gamma_{4\gamma} + \Gamma^\gamma_{\alpha\alpha}\Gamma^4_{4\gamma}\right] +$$

$$+g_{\alpha[\alpha}g_{\gamma]\gamma}\Gamma^\alpha_{44}\Gamma^\gamma_{\alpha\gamma} + g_{\gamma[\gamma}g_{4]4}\Gamma^\gamma_{\alpha\alpha}\Gamma^4_{\gamma 4} \quad , \tag{2.29}$$

$$F_{\alpha\gamma} \equiv \widetilde{g}^{\alpha\alpha}\widetilde{g}^{\gamma\gamma}\widetilde{g}^{\delta_1\delta_1}g_{\gamma[\gamma}g_{\delta_1]\delta_1}\Gamma^\gamma_{\alpha\alpha}\Gamma^{\delta_1}_{\gamma\delta_1} + \widetilde{g}^{\alpha[\alpha}\widetilde{g}^{\gamma]\gamma}[\widetilde{g}^{\delta_1\delta_1}g_{\gamma\gamma}g_{\delta_1\delta_1}\Gamma^{\delta_1}_{\alpha\alpha}\Gamma^\gamma_{\gamma\delta_1} +$$

$$+\Gamma^r_{\alpha\alpha}g_{r\gamma} + \left(\Gamma^\delta_{\alpha\alpha}g_{\delta\delta}\right)_{,\gamma}] \quad . \tag{2.30}$$

In (2.25 - 2.30) the Greek indices run the values $\alpha, \beta, \gamma = 1, 2, 3$, while all the other indices run from 1 to 4. Equation (2.25) can be considered also as a cubic algebraic equation in respect to $\widetilde{g}^{44}, \widetilde{g}^{\alpha\alpha}$ and $\widetilde{g}^{\alpha\alpha}_{,\alpha}$. It is understood also that the connection components are known in advance, but if they are not - then the equation will be no longer a cubic one, but a higher order algebraic equation.

3 EMBEDDED SEQUENCE OF ALGEBRAIC EQUATIONS AND FINDING THE SOLUTIONS OF THE CUBIC ALGEBRAIC EQUATION

The purpose of the present subsection will be to describe the method for finding the solution (i. e . the algebraic variety of the differentials dX^i) of the cubic algebraic equation (2.14) (in the limit $d^2X^k = 0$). The applied method has been proposed first in [10] but here it will be developed further and applied in respect to a **sequence of algebraic equations with algebraic varieties, which are embedded into the initial one. This means that if at first the algorithm is applied in respect to the three-dimensional cubic algebraic equation (2.14) and a solution for** dX^3 (depending on the Weierstrass function and its derivative is found), then **the same algorithm will be applied in respect to the two-dimensional cubic algebraic equation with variables dX^1 and dX^2, and finally to the one-dimensional cubic algebraic equation of the variable dX^1 only.**

The basic knowledge about the parametrization of a cubic algebraic equation with the Weierstrass function and its derivative are given in almost all basic textbooks on elliptic functions [9, 11, 32, 33] and many others. However, the most complete, detailed and exhaustive knowledge about elliptic functions and automorphic forms is contained in the two two - volume books [34, 35] of Felix Klein and Robert Fricke, written more than 100 years ago. More specific and advanced topics on elliptic curves from a mathematical point of view such as the group of rational points, cubic curves over finite fields, families

of elliptic curves and torsion points and etc. are contained in the monographs [36, 37]. A very understandable exposition of the classical topics on cubic algebraic curves and at the same time the most contemporary issues such as the Mordell's and Dirichlet's theorems and L functions, modular forms and theories of Eichler - Shimura are given in the book of Knapp [38], which can be used for first acquintance in these topics. A consistent, modern and full exposition of elliptic curves in the language of modern mathematics is given in the (two consequent) monographs of Silverman [39, 40]. A classical and very understandable exposition of the relation of elliptic curves with modular forms is given in [41], also in [42]. From a modern standpoint the relation of elliptic curves with number theory and modular forms is given in the review articles of Cohen and Don Zagier in [43], also introductory knowledge on hyperelliptic integrals, compact Riemann surfaces and Abelian varieties are presented in the review article by Bost also in [43].

The basic and very simple idea about parametrization of a cubic algebraic equation with the Weierstrass function can be presented as follows: Let us define the lattice $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in Z; \omega_1, \omega_2 \in C, Im\frac{\omega_1}{\omega_2} > 0\}$ and the mapping $f : C/\Lambda \rightarrow CP^2$, which maps the factorized (along the points of the lattice Λ) part of the points on the complex plane into the two **dimensional complex projective space** CP^2. This means that each point z on the complex plane is mapped into the point $(x, y) = (\rho(z), \rho^{'}(z))$, where x and y belong to the **affine curve**

$$y^2 = 4x^3 - g_2x - g_3 \quad , \tag{3.1}$$

where the complex numbers g_2 and g_3 are the so called **Eisenstein series** $g_2 = 60 \sum\limits_{\omega \subset \Gamma} \frac{1}{\omega^4} = \sum\limits_{n,m} \frac{1}{(n+m\tau)^4}$; $g_3 = 140 \sum\limits_{\omega \subset \Gamma} \frac{1}{\omega^6} = \sum\limits_{n,m} \frac{1}{(n+m\tau)^6}$ and $\rho(z)$ denotes the **Weierstrass elliptic function** $\rho(z) = \frac{1}{z^2} + \sum\limits_{\omega} \left[\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right]$ and the summation is over the poles in the complex plane. In other words, the functions $x = \rho(z)$ and $y = \rho^{'}(z)$ are uniformization functions for the cubic curve and it can be proved [9] that the only cubic algebraic curve (but with number coefficients!) which is parametrized by the uniformization functions $x = \rho(z)$ and $y = \rho^{'}(z)$ is the above mentioned affine curve. Note that the fact that the coefficients are numbers is very important, otherwise $\rho(z)$ and $\rho^{'}(z)$ might in principle parametrize a cubic equation of the general type $y^2(z) = a_3(z)x^3 + a_2(z)x^2 + a_1(z)x + a_0(z)$. This problem has also been treated in [10], but still it has numerous unexplored issues, because it is related to the so called **non-arithmetic theory** of algebraic equations, which has not been yet investigated.

In the case of the cubic equation (2.14), the aim will be again to bring the equation to the form (3.1) and afterwards to make equal each of the coefficient functions to the (numerical) coefficients in (2.1).

In order to provide a more clear description of the developed method, let us divide it into several steps.

Step 1. The initial cubic algebraic equation (2.14) is presented as a cubic equation in respect to the variable dx^3 only

$$A_3(dX^3)^3 + B_3(dX^3)^2 + C_3(dX^3) + G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) \equiv 0 \quad , \tag{3.2}$$

where naturally the coefficient functions A_3, B_3 , C_3 and $G^{(2)}$ depend on the variables dX^1 and dX^2 of the algebraic subvariety and on the metric tensor g_{ij}, the Christoffel connection Γ_{ij}^k and the Ricci tensor R_{ij}:

$$A_3 \equiv 2p\Gamma_{33}^r g_{3r} \quad ; \qquad B_3 \equiv 6p\Gamma_{\alpha 3}^r g_{3r} dX^\alpha - R_{33} \quad , \tag{3.3}$$

$$C_3 \equiv -2R_{\alpha 3} dX^\alpha + 2p(\Gamma_{\alpha\beta}^r g_{3r} + 2\Gamma_{3\beta}^r g_{\alpha r}) dX^\alpha dX^\beta \quad . \tag{3.4}$$

The Greek indices α, β take values $\alpha, \beta = 1, 2$ while the indice r takes values $r = 1, 2, 3$.

Step 2. A linear-fractional transformation

$$dx^3 = \frac{a_3(z)\widetilde{dX}^3 + b_3(z)}{c_3(z)\widetilde{dX}^3 + d_3(z)} \tag{3.5}$$

is performed with the purpose of setting up to zero the coefficient functions in front of the highest (third) degree of $\widetilde{dX}^3$. This will be achieved if $G^{(2)}(dX^2, dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) = -\frac{a_3 Q}{c_3^3}$, where

$$Q \equiv A_3 a_3^2 + C_3 c_3^2 + B_3 a_3 c_3 + 2c_3 d_3 C_3 \quad , \tag{3.6}$$

which gives a cubic algebraic equation in respect to the **two-dimensional algebraic variety** of the variables dX^1 and dX^2:

$$p\Gamma_{\gamma(\alpha}^r g_{\beta)r} dX^\gamma dX^\alpha dX^\beta + K_{\alpha\beta}^{(1)} dX^\alpha dX^\beta + K_\alpha^{(2)} dX^\alpha + 2p\left(\frac{a_3}{c_3}\right)^3 \Gamma_{33}^r g_{3r} = 0 \tag{3.7}$$

and $K_{\alpha\beta}^{(1)}$ and $K_\alpha^{(2)}$ are the corresponding quantities [10]

$$K_{\alpha\beta}^{(1)} \equiv -R_{\alpha\beta} + 2p\frac{a_3}{c_3}(1 + 2\frac{d_3}{c_3})(2\Gamma_{\alpha\beta}^r g_{3r} + \Gamma_{3\alpha}^r g_{\beta r}) \tag{3.8}$$

and

$$K_\alpha^{(2)} \equiv 2\frac{a_3}{c_3}\left[3p\frac{a_3}{c_3}\Gamma_{\alpha 3}^r g_{3r} - (1 + 2\frac{d_3}{c_3})R_{\alpha 3}\right] \quad . \tag{3.9}$$

Note that since the linear fractional transformation (with another coefficient functions) will again be applied in respect to another cubic equations (further it will be explained how they are obtained), everywhere in (3.5 - 3.8) the coefficient functions $a_3(z)$, $b_3(z)$, $c_3(z)$ and $d_3(z)$ bear the indice "3" , to distinguish them from the indices in the other linear-fractional tranformations. In terms of the new variable $n_3 = \widetilde{dX}^3$ the original cubic equation (2.14) acquires the form [10]

$$\widetilde{n}^2 = \overline{P}_1(\widetilde{n})\ m^3 + \overline{P}_2(\widetilde{n})\ m^2 + \overline{P}_3(\widetilde{n})\ m + \overline{P}_4(\widetilde{n}) \ , \tag{3.10}$$

where $\overline{P}_1(\widetilde{n})$, $\overline{P}_2(\widetilde{n})$, $\overline{P}_3(\widetilde{n})$ and $\overline{P}_4(\widetilde{n})$ are complicated functions of the ratios $\frac{c_3}{d_3}$, $\frac{b_3}{d_3}$ and A_3, B_3, C_3 (but not of the ratio $\frac{a_3}{d_3}$, which will become evident that is very important).

The variable m denotes the ratio $\frac{a_3}{c_3}$ and the variable $\widetilde{n}$ is related through the variable n through the expresssion

$$\widetilde{n} = \sqrt{k_3}\sqrt{C_3}\left[n + L_1^{(3)}\frac{B_3}{C_3} + L_2^{(3)}\right] \qquad , \tag{3.11}$$

where

$$k_3 \equiv \frac{b_3}{d_3}\frac{c_3}{d_3}(\frac{c_3}{d_3} + 2) \qquad , \tag{3.12}$$

$$L_1^{(3)} \equiv \frac{1}{2}\frac{\frac{b_3}{d_3}}{\frac{c_3}{d_3} + 2} \quad ; \quad L_2^{(3)} \equiv \frac{1}{\frac{c_3}{d_3} + 2} \quad . \tag{3.13}$$

The subscript "3" in $L_1^{(3)}$ and $L_2^{(3)}$ means that the corresponding ratios in the R. H. S. also have the same subscript. Setting up the coefficient functions $\overline{P}_1(\widetilde{n})$,$\overline{P}_2(\widetilde{n})$, $\overline{P}_3(\widetilde{n})$ equal to the number coefficients $4, 0, -g_2, -g_3$ respectively, one can now parametrize the resulting equation

$$\widetilde{n}^2 = 4m^3 - g_2 m - g_3 \tag{3.14}$$

according to the standard prescription

$$\widetilde{n} = \rho^{'}(z) = \frac{d\rho}{dz} \qquad\qquad \frac{a_3}{c_3} \equiv \ m = \rho(z) \quad . \tag{3.15}$$

Taking this into account, representing the linear-fractional transformation (3.5) as (dividing by c_3)

$$dx^3 = \frac{\frac{a_3}{c_3}\widetilde{dX}^3 + \frac{b_3}{c_3}}{\widetilde{dX}^3 + \frac{d_3}{c_3}} \tag{3.16}$$

and combining expressions (3.11) for $\widetilde{n}$ and (3.16), one can obtain the final formulae for dX^3 as a solution of the cubic algebraic equation

$$dX^3 = \frac{\frac{b_3}{c_3} + \frac{\rho(z)\rho^{'}(z)}{\sqrt{k_3}\sqrt{C_3}} - L_1^{(3)}\frac{B_3}{C_3}\rho(z) - L_2^{(3)}\rho(z)}{\frac{d_3}{c_3} + \frac{\rho^{'}(z)}{\sqrt{k_3}\sqrt{C_3}} - L_1^{(3)}\frac{B_3}{C_3} - L_2^{(3)}} \qquad . \tag{3. 17}$$

In order to be more precise, it should be mentioned that the identification of the functions $\overline{P}_1(\widetilde{n})$,$\overline{P}_2(\widetilde{n})$, $\overline{P}_3(\widetilde{n})$ with the number coefficients gives some additional equations [10], which in principle have to be taken into account in the solution for dx^3. This has been investigated to a certain extent in [10], and will be continued to be investigated in the subsequent parts of this paper. It will further be shown that a cubic equation with coefficient functions, that are quadratic (or cubic) polynomials can be parametrized in **two ways**, and it should be required that the two parametrizations are consistent with one another, i.e. they would give one and the same result. Here in this paper the main objective will be to show the dependence of the solutions on the Weierstrass function and its derivative, which from the given coefficient functions in Appendix B is evident that is

very complicated. Since only the ratios $\frac{b}{d}$ and $\frac{c}{d}$ enter these additional relations, and not $\frac{a}{c}$ (which is related to the Weierstrass function), they do not affect the solution in respect to $\rho(z)$ and $\rho^{'}(z)$.

Since B_3 and C_3 depend on dX^1 and dX^2, the solution (3.17) for dX^3 shall be called the embedding solution for dX^1 and dX^2.

Step 3. Let us now consider the two-dimensional cubic equation (3.7). Following the same approach and finding the "reduced" cubic algebraic equation for dX^1 only, it shall be proved that the solution for dX^2 is the embedding solution for dX^1.

For the purpose, let us again write down eq. (3.7) in the form (3.2), singling out the variable dX^2:

$$A_2(dX^2)^3 + B_2(dX^2)^2 + C_2(dX^2) + G^{(1)}(dX^1, g_{ij}, \Gamma_{ij}^k, R_{ik}) \equiv 0 \qquad , \tag{3.18}$$

where the coefficient functions A_2, B_2, C_2 and $G^{(1)}$ are the following:

$$A_2 \equiv 2p\Gamma_{22}^r g_{2r} \quad ; \qquad B_2 \equiv K_{22}^{(1)} + 2p[2\Gamma_{12}^r g_{2r} + \Gamma_{22}^r g_{1r}]dX^1 \quad , \tag{3.19}$$

$$C_2 \equiv 2p[\Gamma_{11}^r g_{2r} + 2\Gamma_{12}^r g_{1r})(dX^1)^2 + (K_{12}^{(1)} + K_{21}^{(1)})dX^1 + K_2^{(2)} \quad , \tag{3.20}$$

$$G^1 \equiv 2p\Gamma_{11}^r g_{1r}(dX^1)^3 + K_{11}^{(1)}(dX^1)^2 + K_1^{(2)}dX^1 + 2p\rho^3(z)\Gamma_{33}^r g_{3r} \quad . \tag{3.21}$$

Note that the starting equation (3.7) has the same structure of the first terms, if one makes the formal substitution $-R_{\alpha\beta} \rightarrow K_{\alpha\beta}^{(1)}$ in the second terms, but eq. (3.7) has two more additional terms $K_1^{(2)}dX^1 + 2p\rho^3(z)\Gamma_{33}^r g_{3r}$. Therefore, one might guess how the coefficient functions will look like just by taking into account the above substitution and the contributions from the additional terms. Revealing the general structure of the coefficient functions might be particularly uiseful in higher dimensions, when one would have a "chain" of cubic algebraic equations, each of which could be written in the form (3.18) with the prescribed coefficient functions. This is also an interesting and probably not too easy problem for future investigation - is it possible to write down the coefficient functions for arbitrary dimensions. Concretely for the three-dimensional case, investigated here, C_2 in (3.20) can be obtained from C_3 in (3.4), observing that there will be an additional contribution from the term $K_\alpha^{(2)}dX^\alpha$ for $\alpha = 2$. Also, in writing down the coefficient function (3.2) it has been accounted that as a result of the previous parametrization $\frac{a_3}{c_3} = \rho(z)$.

Since eq. (3.18) is of the same kind as eq. (3.2), for which we already wrote down the solution, the expression for dX^2 will be of the same kind as in formulae (3.17), but with the corresponding functions A_2, B_2, C_2 instead of A_3, B_3, C_3. Taking into account (3.19 - 3.20), the solution for dX^2 can be written as follows:

$$dX^2 = \frac{\frac{1}{\sqrt{k_2}}\rho(z)\rho^{'}(z)\sqrt{C_2} + h_1(dX^1)^2 + h_2(dX^1) + h_3}{\frac{1}{\sqrt{k_2}}\rho^{'}(z)\sqrt{C_2} + l_1(dX^1)^2 + l_2(dX^1) + l_3} \qquad , \tag{3.22}$$

where $h_1, h_2, h_3, l_1, l_2, l_3$ are expressions, depending on $\frac{b_2}{d_2}, \frac{d_2}{c_2}, \Gamma_{\alpha\beta}^r$ $(r = 1,2,3\ ;\ \alpha,\beta = 1,2)$, $g_{\alpha\beta}$, $K_{12}^{(1)}$, $K_{21}^{(1)}$ and on the Weierstrass function. They will be presented in Appendix B.

The representation of the solution for dX^2 in the form (3.22) shows that it is an embedding solution of dX^1.

Step 4. It remains now to investigate the **one-dimensional** cubic algebraic equation

$$A_1(dx^1)^3 + B_1(dx^1)^2 + C_1(dx^1) + G^{(0)}(g_{ij}, \Gamma_{ij}^k, R_{ik}) \equiv 0 \qquad , \tag{3.23}$$

obtained from the two-dimensional cubic algebraic equation (3.18) after applying the linear-fractional transformation

$$dx^2 = \frac{a_2(z)\widetilde{dX}^2 + b_2(z)}{c_2(z)\widetilde{dX}^2 + d_2(z)} \tag{3.24}$$

and setting up to zero the coefficient function before the highest (third) degree of $(dX^2)^3$. Taking into account that as a result of the previous parametrization $\frac{a_2}{c_2} = \rho(z)$, the coefficient functions $A_1, B_1, C_1$and D_1 are given in a form, not depending on dX^2 and dX^3:

$$A_1 \equiv 2p\Gamma_{11}^r g_{1r} \qquad , \tag{3.25}$$

$$B_1 \equiv F_3\rho(z) + K_{11}^{(1)} = 2p(1 + 2\frac{d_2}{c_2})[2\Gamma_{12}^r g_{1r} + \Gamma_{11}^r g_{2r}]\rho(z) + K_{11}^{(1)} \qquad , \tag{3.26}$$

$$C_1 \equiv F_1\rho^2(z) + F_2\rho(z) + K_1^{(2)} = 2p[2\Gamma_{12}^r g_{2r} + \Gamma_{22}^r g_{1r}]\rho^2(z) +$$
$$+(1 + 2\frac{d_2}{c_2})(K_{12}^{(1)} + K_{21}^{(1)})\rho(z) + K_1^{(2)} \qquad , \tag{3.27}$$

$$G^0 \equiv 2p[\Gamma_{22}^r g_{2r} + \Gamma_{33}^r g_{3r}]\rho^3(z) + K_{22}^{(1)}\rho^2(z) \qquad . \tag{3.28}$$

The solution for dX^1 can again be written in the form (3.17), but with $\frac{b_1}{c_1}$, $\frac{d_1}{c_1}$, $L_1^{(1)}$, $L_2^{(1)}$, $k_1$and B_1, C_1 instead of these expressions with the indice "3".

Taking into account formulaes (3.25 - 3.28) for A_1, B_1 and C_1, the final expression for dx^1 can be written as

$$dX^1 = \frac{\frac{1}{\sqrt{k_1}}\rho(z)\rho'(z)\sqrt{F_1\rho^2 + F_2\rho(z) + K_1^{(2)}} + f_1\rho^3 + f_2\rho^2 + f_3\rho + f_4}{\frac{1}{\sqrt{k_1}}\rho'(z)\sqrt{F_1\rho^2(z) + F_2\rho(z) + K_1^{(2)}} + \widetilde{g}_1\rho^2(z) + \widetilde{g}_2\rho(z) + \widetilde{g}_3} \qquad , \tag{3.29}$$

where $F_1, F_2, f_1, f_2, f_3, f_4, \widetilde{g}_1, \widetilde{g}_2$ and $\widetilde{g}_3$ are functions (also to be given in Appendix B), depending on $g_{\alpha\beta}$, $\Gamma_{\alpha\beta}^r$ $(\alpha, \beta = 1, 2)$ and on the ratios $\frac{b_1}{c_1}$, $\frac{b_1}{d_1}$, $\frac{b_2}{d_2}$, $\frac{d_1}{c_1}$, $\frac{d_2}{c_2}$.

4 A PROOF THAT THE SOLUTIONS dX^1, dX^2 AND dX^3 ARE NOT ELLIPTIC FUNCTIONS

Proposition 1 *The expressions (3.22) for dX^2 and (3. 29) for dX^1* ***do not*** *represent elliptic functions.*

Proof: The proof is straightforward and will be based on assuming the contrary. Let us first assume that dX^1 is an elliptic function. Then from standard theory of elliptic functions it follows that dX^1(being an elliptic function by assumption) can be represented as

$$dX^1 = K_1(\rho) + \rho^{'}(z)K_2(\rho) \quad , \tag{4.1}$$

where $K_1(\rho)$ and $K_2(\rho)$ depend on the Weierstrass function only. For convenience one may denote the expressions outside the square root in the nominator and denominator as

$$Z_1(\rho) \equiv f_1\rho^3(z) + f_2\rho^2(z) + f_3\rho(z) + f_4 \quad , \tag{4.2}$$

$$Z_2(\rho) \equiv g_1\rho^2(z) + g_2\rho(z) + g_3 \quad . \tag{4. 3}$$

Then, setting up equal the expressions (4.1) and (3.29) for dX^1, one can express the function $K_2(\rho)$ as

$$K_2(\rho) = \frac{(1-K_1(\rho))\rho\rho^{'}\sqrt{F_1\rho^2 + F_2\rho + K_1^{(2)}} + \sqrt{k_1}Z_1(\rho) - Z_2(\rho)K_1(\rho)}{\rho^{'}\left[\rho\sqrt{F_1\rho^2 + F_2\rho + K_1^{(2)}} + \sqrt{k_1}Z_2(\rho)\right]} \quad . \tag{4.4}$$

The R. H. S. of the above expression depends on the derivative $\rho^{'}$, while the L.H. S. depends on ρ only. Therefore the obtained contradiction is due to the initial assumption that dX^1 is an elliptic function.

In order to prove that dX^2 is not an elliptic function, let us first observe that (3.22) can be solved in respect to dX^1 as a cubic algebraic equation by means of the Wiet formulae. For the purpose, first the variable change

$$dX^1 = dy^1 - \frac{B_1}{3} \tag{4.5}$$

should be performed, after which equation (3.23) assumes the form

$$\overline{A}_1(dy^1)^3 + \overline{B}_1(dy^1) + \overline{C}_1 = 0 \quad . \tag{4. 6}$$

The solution of (4.6) is given by

$$dy^1 = \sqrt[3]{-\frac{\overline{C}_1}{2\overline{A}_1} - \sqrt[2]{\frac{1}{4}\left(\frac{\overline{C}_1}{\overline{A}_1}\right)^2 + \frac{1}{27}\left(\frac{\overline{B}_1}{\overline{A}_1}\right)^3}} +$$

$$+\sqrt[3]{-\frac{\overline{C}_1}{2\overline{A}_1} + \sqrt[2]{\frac{1}{4}\left(\frac{\overline{C}_1}{\overline{A}_1}\right)^2 + \frac{1}{27}\left(\frac{\overline{B}_1}{\overline{A}_1}\right)^3}} \quad , \tag{4. 7}$$

where $\overline{A}_1, \overline{B}_1$ and $\overline{C}_1$ depend on the coefficient functions A_1, B_1 and C_1:

$$\overline{A}_1 = A_1 \quad ; \quad \overline{B}_1 = \frac{A_1 B_1^2}{3} - \frac{2B_1^2}{3} + C_1 \quad , \tag{4. 8}$$

$$\overline{C}_1 = -\frac{A_1 B_1^3}{27} + \frac{B_1^3}{9} - \frac{C_1 B_1}{3} + G^0 \quad . \tag{4. 9}$$

The important conclusion from the above formulaes (4. 5 - 4. 9) is that dX^1 is an **irrational function**, depending on A_1, B_1, C_1 and therefore only on $g_{\alpha\beta}$ and $\Gamma^r_{\alpha\beta}$. Therefore

$$dX^1 = O_1(g_{\alpha\beta}, \Gamma^r_{\alpha\beta}) \quad . \tag{4. 10}$$

In the same manner by assuming the contrary, it can be proved that dX^2 is not an elliptic function.

Now it can be seen that the two equations (3. 22) and (3. 29) for dX^2 and dX^1represent a complicated relation between $g_{\alpha\beta}, \Gamma^r_{\alpha\beta}$ and the Weierstrass function. Unfortunately, this relation cannot be found explicitely, perhaps for some very simple choices of the metric. At least, it can be pointed out how it can be found. For example, from (4. 4) for $dx^1 = O_1(g_{\alpha\beta}, \Gamma^r_{\alpha\beta}) = K_1(\rho)$ and $K_2(\rho) = 0$ the following relation can be obtained

$$\int \frac{(1-O_1)\rho\sqrt{F_1\rho^2 + F_2\rho + K_1^{(2)}}}{Z_2(\rho)O_1 - \sqrt{k_1}Z_1(\rho)} d\rho = \int dz \quad , \tag{4. 11}$$

from where, if the integration can be performed, it can be found

$$F(\rho) = z + const. \tag{4. 12}$$

Let us further take the second relation (3. 22) for dX^2 for the value (4.10) of $dX^1 = O_1(g_{\alpha\beta}, \Gamma^r_{\alpha\beta})$.The R. H. S. of (3. 22) can be represented as

$$dX^2 = \frac{\frac{1}{\sqrt{k_2}}\rho\rho^{'}\sqrt{C_2(O_1, g_{\alpha\beta}, \Gamma^r_{\alpha\beta})} + K_1(O_1, g_{\alpha\beta}, \Gamma^r_{\alpha\beta})}{\frac{1}{\sqrt{k_2}}\rho^{'}\sqrt{C_2} + K_2(O_1, g_{\alpha\beta}, \Gamma^r_{\alpha\beta})} \quad . \tag{4. 13}$$

At the same time, if expression (3. 29) for dX^1 is substituted into the formulae (3. 22) for dX^2, one obtains another expression, where the functions $\overline{C}_2, \overline{K}_1$ and $\overline{K}_2$ will depend will depend additionally on a complicated manner on ρ and $\rho^{'}$:

$$dX^2 = \frac{\frac{1}{\sqrt{k_2}}\rho\rho^{'}\sqrt{\overline{C}_2(O_1, g_{\alpha\beta}, \Gamma^r_{\alpha\beta}, \rho, \rho^{'})} + \overline{K}_1(O_1, g_{\alpha\beta}, \Gamma^r_{\alpha\beta}, \rho, \rho^{'})}{\frac{1}{\sqrt{k_2}}\rho^{'}\sqrt{\overline{C}_2} + \overline{K}_2(O_1, g_{\alpha\beta}, \Gamma^r_{\alpha\beta}, \rho, \rho^{'})} \quad . \tag{4. 14}$$

Therefore, setting up equal (4.13) and (4. 14), differentiating (4.12) by z in order to get $\rho^{'} = \frac{1}{F^{'}(\rho)}$ and substituting $\rho^{'}$ in the R. H. S. of (4. 14), one obtains an expression of the kind:

$$G(g_{\alpha\beta}, \Gamma^r_{\alpha\beta}, \rho) = 0 \quad . \tag{4. 15}$$

5 COMPLEX COORDINATE DEPENDENCE OF THE METRIC TENSOR COMPONENTS FROM THE UNIFORMIZATION OF A CUBIC ALGEBRAIC SURFACE

In section 2 it was asserted that the covariant and contravariant metric tensor components constitute an algebraic variety of the solutions of the ninth - degree algebraic equation (2.22).

In this Section it will be shown that the solutions (3. 17), (3. 22) and (3. 29) of the cubic algebraic equation (2.14) enable us to express not only the contravariant metric tensor components through the Weierstrass function and its derivatives, but the covariant components as well.

Let us write down for convenience the system of equations (3. 17), (3. 22) and (3. 29) for dX^1, dX^2 and dX^3 as

$$dX^1(X^1,X^2,X^3) = F_1(g_{ij}(\mathbf{X}),\Gamma^k_{ij}(\mathbf{X}),\rho(z),\rho^{'}(z)) = F_1(\mathbf{X},z) \quad , \tag{5.1}$$

$$dX^2(X^1,X^2,X^3) = F_2(g_{ij}(\mathbf{X}),\Gamma^k_{ij}(\mathbf{X}),\rho(z),\rho^{'}(z)) = F_2(\mathbf{X},z) \quad , \tag{5.2}$$

$$dX^3(X^1,X^2,X^3) = F_3(g_{ij}(\mathbf{X}),\Gamma^k_{ij}(\mathbf{X}),\rho(z),\rho^{'}(z)) = F_3(\mathbf{X},z) \quad , \tag{5.3}$$

where the appearence of the complex coordinate z is a natural consequence of the uniformization procedure, applied in respect to each one of the cubic equations from the "embedded" sequence of equations. Let us put this statement in a more clear way. **Expressions (5.1 - 5.3) can be treated as the uniformization functions for the multivariable cubic algebraic equation (algebraic surface) (2.14).** In other words, it is possible to find uniformization functions not only for 'parametrizable" form $y^2 = 4x^3 - g_2x - g_3$ of a two - dimensional cubic algebraic equation, but also in higher -dimensional case. This opens new possibilities to investigate multicomponent cubic algebraic equations (surfaces), even in the framework of the standard arithmetical theory of algebraic equations.

Yet how the appearence of the additional complex coordinate z on the R. H. S. of (5.1 - 5.3) can be reconciled with the dependence of the differentials on the L. H. S. only on the generalized coordinates (X^1,X^2,X^3) (and on the initial coordinates x^1,x^2,x^3 because of the mapping $X^i = X^i(x^1,x^2,x^3)$)? The only reasonable assumption will be that the initial coordinates depend also on the complex coordinate, i.e.

$$X^1 \equiv X^1(x^1(z),x^2(z),x^3(z)) = X^1(\mathbf{x},\, z) \quad , \tag{5.4}$$

$$X^2 \equiv X^2(x^1(z),x^2(z),x^3(z)) = X^2(\mathbf{x},\, z) \quad , \tag{5.5}$$

$$X^1 \equiv X^3(x^1(z),x^2(z),x^3(z)) = X^3(\mathbf{x},\, z) \quad , \tag{5.6}$$

and thus the generalized coordinates are defined on a Riemann surface. Taking into account the important initial assumptions

$$d^2X^1 = 0 = dF_1(\mathbf{X}(z), z) = \frac{dF_1}{dz}dz \quad , \tag{5.7}$$

$$d^2X^2 = 0 = dF_2(\mathbf{X}(z), z) = \frac{dF_2}{dz}dz \quad , \tag{5.8}$$

$$d^2X^3 = 0 = dF_3(\mathbf{X}(z), z) = \frac{dF_3}{dz}dz \quad , \tag{5.9}$$

one easily gets the system of three inhomogeneous linear algebraic equations in respect to the functions $\frac{\partial X^1}{\partial z}$, $\frac{\partial X^2}{\partial z}$ and $\frac{\partial X^3}{\partial z}$:

$$\frac{\partial F_1}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_1}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_1}{\partial X^3}\frac{\partial X^3}{\partial z} + \frac{\partial F_1}{\partial z} = 0 \quad , \tag{5.10}$$

$$\frac{\partial F_2}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_2}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_2}{\partial X^3}\frac{\partial X^3}{\partial z} + \frac{\partial F_2}{\partial z} = 0 \quad , \tag{5.11}$$

$$\frac{\partial F_3}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_3}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_3}{\partial X^3}\frac{\partial X^3}{\partial z} + \frac{\partial F_3}{\partial z} = 0 \quad . \tag{5.12}$$

The solution of this algebraic system $(i, k = 1, 2, 3)$

$$\frac{\partial X^1}{\partial z} = G_1\left(\frac{\partial F_i}{\partial X^k}\right) = G_1\left(X^1, X^2, X^3, z\right) \quad , \tag{5.13}$$

$$\frac{\partial X^2}{\partial z} = G_2\left(\frac{\partial F_i}{\partial X^k}\right) = G_2\left(X^1, X^2, X^3, z\right) \quad , \tag{5.14}$$

$$\frac{\partial X^3}{\partial z} = G_3\left(\frac{\partial F_i}{\partial X^k}\right) = G_3\left(X^1, X^2, X^3, z\right) \tag{5.15}$$

represents a system of **three first - order nonlinear differential equations.** A solution of this system can always be found in the form

$$X^1 = X^1(z) \quad ; \quad X^2 = X^2(z) \quad ; \quad X^3 = X^3(z) \quad . \tag{5.16}$$

and therefore, the metric tensor components will also depend on the complex coordinate z, i.e. $g_{ij} = g_{ij}(\mathbf{X}(z))$. Note that since the functions $\frac{\partial F_i}{\partial X^k}$ in the R. H. S. of (5.13) - (5.15) depend on the Weierstrass function and its derivatives, it might seem natural to write that **the solution of the above system of nonlinear differential equations** g_{ij} **will also depend on the Weierstrass function and its derivatives**

$$g_{ij} = g_{ij}(X^1(\rho(z), \rho^{'}(z), X^2(\rho(z), \rho^{'}(z), X^3(\rho(z), \rho^{'}(z)) = g_{ij}(z) \quad . \tag{5.17}$$

Note however that for the moment we do not have a theorem that the solution of the system (5.13 - 5.15) will also contain the Weierstrass function.

Now let us mention the other equations, which will further be taken into account.

The first set of equations simply means that the differentials dF_1, dF_2, dF_3, equal to the second differentials d^2X^1, d^2X^2, d^2X^3 can be taken in respect both to the generalized coordinates X^1, X^2, X^3 and the initial coordinates x^1, x^2, x^3

$$d^2X^1 = dF_1(\mathbf{X}(z), z) = dF_1(\mathbf{x}(z), z) \quad , \tag{5.18}$$

$$d^2X^2 = dF_2(\mathbf{X}(z), z) = dF_2(\mathbf{x}(z), z) \quad , \tag{5.19}$$

$$d^2X^3 = dF_3(\mathbf{X}(z), z) = dF_3(\mathbf{x}(z), z) \quad . \tag{5.20}$$

For completeness, the formal proof that the second order differentials can be expressed in different sets of coordinates will be given in Appendix A. Denoting further $\dot{x}^1 \equiv \frac{\partial x^1}{\partial z}$, $\dot{x}^2 \equiv \frac{\partial x^2}{\partial z}$ and $\dot{x}^3 \equiv \frac{\partial x^3}{\partial z}$, the above equalities result again in a system of three inhomogeneous algebraic equations in respect to $\dot{X}^1 \equiv \frac{\partial X^1}{\partial z}$, $\dot{X}^2 \equiv \frac{\partial X^2}{\partial z}$ and $\dot{X}^3 \equiv \frac{\partial X^3}{\partial z}$

$$\frac{\partial F_1}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_1}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_1}{\partial X^3}\frac{\partial X^3}{\partial z} = \frac{\partial F_1}{\partial x^1}\dot{x}^1 + \frac{\partial F_1}{\partial x^2}\dot{x}^2 + \frac{\partial F_1}{\partial x^3}\dot{x}^3 \quad , \tag{5.21}$$

$$\frac{\partial F_2}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_2}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_2}{\partial X^3}\frac{\partial X^3}{\partial z} = \frac{\partial F_2}{\partial x^1}\dot{x}^1 + \frac{\partial F_2}{\partial x^2}\dot{x}^2 + \frac{\partial F_2}{\partial x^3}\dot{x}^3 \quad , \tag{5.22}$$

$$\frac{\partial F_3}{\partial X^1}\frac{\partial X^1}{\partial z} + \frac{\partial F_3}{\partial X^2}\frac{\partial X^2}{\partial z} + \frac{\partial F_3}{\partial X^3}\frac{\partial X^3}{\partial z} = \frac{\partial F_3}{\partial x^1}\dot{x}^1 + \frac{\partial F_3}{\partial x^2}\dot{x}^2 + \frac{\partial F_3}{\partial x^3}\dot{x}^3 \quad . \tag{5.23}$$

Assuming for the moment that we know the functions $\dot{x}^1$, $\dot{x}^2$ and $\dot{x}^3$, the solutions of this algebraic system will give again another system of three first - order nonlinear differential equations

$$\frac{\partial X^1}{\partial z} = H_1\left(X^1, X^2, X^3,\ z\ ,\ \dot{x}^1, \dot{x}^2, \dot{x}^3\right) \quad , \tag{5.24}$$

$$\frac{\partial X^2}{\partial z} = H_2\left(X^1, X^2, X^3,\ z\ ,\ \dot{x}^1, \dot{x}^2, \dot{x}^3\right) \quad , \tag{5.25}$$

$$\frac{\partial X^3}{\partial z} = H_3\left(X^1, X^2, X^3,\ z\ , \dot{x}^1, \dot{x}^2, \dot{x}^3\right) \quad . \tag{5.26}$$

Again, a solution of this system like the one in (5.16) can be obtained but with account of the dependence on the additional variables $\dot{x}^1$, $\dot{x}^2$ and $\dot{x}^3$. Let us also here note that the solution (5.16) of the nonlinear system of equations (5.13 - 5. 15) can be assumed to be dependent on some another complex variable v

$$X^1 = X^1(z, v) \quad ; \quad X^1 = X^1(z, v) \quad ; \quad X^1 = X^1(z, v) \quad . \tag{5.27}$$

The system of equations (5.18 - 5.20) $(i = 1, 2, 3)$

$$d^2X^i = dF_i(\mathbf{X}(z, v), z) = dF_1(\mathbf{x}(z, v), z) \quad , \tag{5.28}$$

with account of the expressions (5.24 - 5.26) now will be rewritten as

$$\frac{\partial F_i}{\partial X^1}\frac{\partial X^1}{\partial v}+\frac{\partial F_i}{\partial X^2}\frac{\partial X^2}{\partial v}+\frac{\partial F_i}{\partial X^3}\frac{\partial X^3}{\partial v}=\frac{\partial F_i}{\partial x^1}\dot{x}^1+\frac{\partial F_i}{\partial x^2}\dot{x}^2+\frac{\partial F_i}{\partial x^3}\dot{x}^3+$$

$$+\frac{\partial F_i}{\partial x^1}x^{'1}+\frac{\partial F_i}{\partial x^2}x^{'2}+\frac{\partial F_i}{\partial x^3}x^{'3}-\frac{\partial F_i}{\partial X^1}H_1-\frac{\partial F_i}{\partial X^2}H_2-\frac{\partial F_i}{\partial X^3}H_3 \quad , \tag{5. 29}$$

where $x^{'1}, x^{'2}, x^{'3}$ denote the derivatives $\frac{\partial x^1}{\partial z}, \frac{\partial x^2}{\partial z}, \frac{\partial x^3}{\partial z}$. The same notation further will be used in respect to the variables $\frac{\partial X^1}{\partial v}, \frac{\partial X^2}{\partial v}, \frac{\partial X^3}{\partial v}$. Similarly to (5. 24) - (5.26), the algebraic solution of this system of equations can be represented as

$$\frac{\partial X^i}{\partial v}=K_i\left(\mathbf{X}(z,v),\ z\ ,\dot{\mathbf{x}}\ ,\mathbf{x}^{'}\right) \quad . \tag{5.30}$$

Note that instead of (5.28), we could have also written

$$d^2X^i=dF_i(\mathbf{X}(z,v),z)=dF_1(\mathbf{x}(z),z,v) \quad . \tag{5.31}$$

Further in section 7 it shall be proved why this would be incorrect. The complete analysis of the system of equations, when both system of coordinates depend on the two pair of complex variables z and v will be given in the following sections. For the moment we give just the general qualitative motivations.

The other set of equations, which will further be used and which relates the generalized coordinates X^i to the initial ones x^i is

$$d^2X^i=0=\frac{\partial^2 X^i}{\partial x^k\partial x^r}dx^k dx^r+\frac{\partial X^i}{\partial x^k}d^2x^k \quad . \tag{5.32}$$

For the moment we assume that the initial coordinates x^k depend only on the z coordinate, and therefore

$$\frac{\partial^2 X^i}{\partial x^k\partial x^r}=\frac{\ddot{X}^i}{\dot{x}^k\dot{x}^r}-\dot{X}^i\frac{\ddot{x}^r}{\dot{x}^k\left(\dot{x}^r\right)^2} \quad . \tag{5.33}$$

Taking this into account, the system (5.32) in the $n-$dimensional case can be written as

$$n^2\ddot{X}^i(dz)^2-(n-1)\dot{X}^i\frac{\ddot{x}^r}{\dot{x}^r}(dz)^2+n\dot{X}^i d^2z=0 \quad . \tag{5.34}$$

Introducing the notation

$$y^r=\frac{\partial}{\partial z}\left(\ln\dot{x}^r\right)=\frac{\ddot{x}^r}{\dot{x}^r} \quad , \tag{5.35}$$

for the three-dimensional case the system (5.34) can be written as

$$2\dot{X}^i(dz)^2(y^1+y^2+y^3)=9\ddot{X}^i(dz)^2+3\dot{X}^i d^2z \quad . \tag{5.36}$$

Dividing the L. H. S. and the R. H. S. of the i-th and the j-th equation of this system, it can easily be obtained

$$(dz)^2 \left(\ddot{X}^i \dot{X}^j - \ddot{X}^j \dot{X}^i \right) = 0 \qquad , \tag{5.37}$$

which with account of the obvious relation

$$\frac{\partial}{\partial z} \left(\frac{\dot{X}^i}{\dot{X}^j} \right) = \frac{\left(\ddot{X}^i \dot{X}^j - \ddot{X}^j \dot{X}^i \right)}{\left(\dot{X}^j \right)^2} \tag{5.38}$$

can be written as

$$\left(\dot{X}^j \right)^2 \frac{\partial}{\partial z} \left(\frac{\dot{X}^i}{\dot{X}^j} \right) = 0 \qquad . \tag{5.39}$$

Neglecting the case when $\dot{X}^j = 0$, the above relation simply means that $\dot{X}^2$ and $\dot{X}^3$ should be proportional to $\dot{X}^1$

$$\dot{X}^2 = C_2 \dot{X}^1 \quad ; \quad \dot{X}^3 = C_3 \dot{X}^1 \qquad , \tag{5.40}$$

where C_2 and C_3 are constants. Indeed, it is easily seen that (5.40) holds since

$$dX^1 = \dot{X}^1 dz = F_1 \ ; \quad dX^2 = \dot{X}^2 dz = F_2 \ ; \quad dX^3 = \dot{X}^3 dz = F_3 \tag{5.41}$$

and consequently

$$C_2 = \frac{F_2}{F_1} \qquad ; \qquad C_3 = \frac{F_3}{F_1} \qquad . \tag{5.42}$$

6 FIRST-ORDER NONLINEAR DIFFERENTIAL EQUATIONS FOR THE COMPLEX FUNCTIONS $x = x(z)$ AND $X = X(z)$

For the purpose, the two systems of algebraic equations (5.10) - (5.12) and (5.21) - (5.23) will be used. If one substitutes the found expressions (5.40) for $\dot{X}^2$ and $\dot{X}^3$ into the system (5.10) - (5.12), one may treat it as an algebraic system of equations in respect to the variables $\dot{X}^1$, C_2 and C_3. Introducing the notation

$$\{F_i, F_j\}_{z,X^k} \equiv \frac{\partial F_i}{\partial z} \frac{\partial F_j}{\partial X^k} - \frac{\partial F_i}{\partial X^k} \frac{\partial F_j}{\partial z} \tag{6.1}$$

for the **"one-dimensional" Poisson bracket** $\{F_i, F_j\}_{z,X^k}$ of the coordinates z, X^k and also the notation

$$\{F_1, F_2, F_3\}_{z,[X^i,X^j]} \equiv \{F_1, F_2\}_{z,X^i}\{F_1, F_3\}_{z,X^j} - \{F_1, F_2\}_{z,X^j}\{F_1, F_3\}_{z,X^i} \quad , \tag{6.2}$$

one can show that the solution of the system of linear algebraic equations (5.10)-(5.12) in respect to $\dot{X}^1, C_2$ and C_3

$$\frac{\partial F_i}{\partial X^1}\dot{X}^1 + \frac{\partial F_i}{\partial X^2}\dot{X}^2 + \frac{\partial F_i}{\partial X^3}\dot{X}^3 + \frac{\partial F_i}{\partial z} = 0 \tag{6.3}$$

can be represented in the following compact form

$$C_2 = \frac{\{F_1, F_2, F_3\}_{z,[X^3,X^1]}}{\{F_1, F_2, F_3\}_{z,[X^2,X^3]}} \quad ; \quad C_3 = \frac{\{F_1, F_2, F_3\}_{z,[X^1,X^2]}}{\{F_1, F_2, F_3\}_{z,[X^2,X^3]}} \quad , \tag{6.4}$$

$$\dot{X}^1 = -\frac{\frac{\partial F_1}{\partial z}\{F_1, F_2, F_3\}_{z,[X^2,X^3]}}{K_1} \quad . \tag{6.5}$$

In (6.5) the following notation has been introduced for K_i $(i = 1, 2, 3)$

$$K_i \equiv \frac{\partial F_i}{\partial X^1}\{F_1, F_2, F_3\}_{z,[X^2,X^3]} + \frac{\partial F_i}{\partial X^2}\{F_1, F_2, F_3\}_{z,[X^3,X^1]} + \frac{\partial F_i}{\partial X^3}\{F_1, F_2, F_3\}_{z,[X^1,X^2]} \quad . \tag{6.6}$$

The usefulness of introducing this notation will soon be understood.

Now let us rewrite the system of equations (5.21) - (5.23) in the form

$$\frac{\partial F_i}{\partial x^1}\dot{x}^1 + \frac{\partial F_i}{\partial x^2}\dot{x}^2 + \frac{\partial F_i}{\partial x^3}\dot{x}^3 = M_i \quad , \tag{6.7}$$

where M_i will be the notation for

$$M_i \equiv \frac{\partial F_i}{\partial X^1}\dot{X}^1 + \frac{\partial F_i}{\partial X^2}\dot{X}^2 + \frac{\partial F_i}{\partial X^3}\dot{X}^3 \quad . \tag{6.8}$$

Making use of the above formulaes (6.4 - 6.6) and also (5.40), M_i can be calculated to be

$$M_i = -\frac{\partial F_1}{\partial z}\frac{K_i}{K_1} \quad . \tag{6.9}$$

Further, the solutions of the linear algebraic system of equations (6.7) can be represented in the form

$$\dot{x}^i = S_1^i M_1 + S_2^i M_2 + S_3^i M_3 \tag{6.10}$$

where the functions S_1^i, S_2^i and S_3^i depend on $\frac{\partial F_i}{\partial x^k}$ $(i, k = 1, 2, 3)$. Since $M_1,\ M_2,\ M_3$ according to (6.9) and (6.6) are proportional to $\frac{\partial F_1}{\partial z}\frac{\{F_1,F_2,F_3\}_{z,[X^k,X^j]}}{K_1}$ (where $(k, j) = (2, 3)$, $(3, 1)$ or $(1, 2)$), the resulting solution (6.10) will be of the kind

$$\dot{x}^i = \frac{\overline{S}_1^i\left(\frac{\partial F_1}{\partial z}\right)^2 + \overline{S}_2^i\frac{\partial F_2}{\partial z}\frac{\partial F_1}{\partial z} + \overline{S}_3^i\frac{\partial F_1}{\partial z}\frac{\partial F_3}{\partial z}}{\overline{S}_4^i\left(\frac{\partial F_1}{\partial z}\right) + \overline{S}_5^i\frac{\partial F_2}{\partial z} + \overline{S}_6^i\frac{\partial F_3}{\partial z}} \quad , \tag{6.11}$$

where the functions $\overline{S}_1^i, \overline{S}_2^i, ..., \overline{S}_6^i$ depend both on $\frac{\partial F_i}{\partial X^k}$ and $\frac{\partial F_i}{\partial x^k}$ and consequently on all the variables x^k, X^k and z. We have used also the following relation, obtained after simple algebra with account of (6.1) and (6.2)

$$\{F_1, F_2, F_3\}_{z,[X^i,X^j]} = \left(\frac{\partial F_1}{\partial z}\right)^2 \{F_2, F_3\}_{X^i,X^j} +$$

$$+\frac{\partial F_1}{\partial z}\frac{\partial F_2}{\partial z}\{F_3, F_1\}_{X^i,X^j} + \frac{\partial F_1}{\partial z}\frac{\partial F_3}{\partial z}\{F_1, F_2\}_{X^i,X^j} \quad . \tag{6.12}$$

Thus we have obtained the system of first order nonlinear differential equations in respect to the initial coordinates $x^i = x^i(z)$. An analogous system of nonlinear differential equations is obtained for $X^1 = X^1(z)$, $X^2 = X^2(z)$ and $X^3 = X^3(z)$ - for X^1 this is equation (6.5), and with aaccount of (5.40) and expressions (6.4) for C_2 and C_3, the corresponding equations for $X^2(z)$ and $X^3(z)$ are

$$\dot{X}^2 = -\frac{\frac{\partial F_1}{\partial z}\{F_1, F_2, F_3\}_{z,[X^3,X^1]}}{K_1} \quad ; \quad \dot{X}^1 = -\frac{\frac{\partial F_1}{\partial z}\{F_1, F_2, F_3\}_{z,[X^1,X^2]}}{K_1} \quad . \tag{6.13}$$

Therefore, if the generalized coordinates X^1, X^2, X^3 are determined as functions of the complex variable z after solving the system (6.5), (6.13), the obtained functions $X^1 = X^1(z)$, $X^2 = X^2(z)$ and $X^3 = X^3(z)$ can be substituted into the R. H. S. of the system (6.11) for x^1, x^2 and x^3 and the corresponding solutions $x^1 = x^1(z)$, $x^2 = x^2(z)$ and $x^3 = x^3(z)$ can be found. Remember that we started from the assumption that only the generalized coordinates X^1, X^2, X^3 satisfy the original cubic algebraic equation and therefore equalities (5.41) are fulfilled. **Nevertheless, in spite of the fact that it had not been assumed that the initial coordinates satisfy the equations $dx^i = F_i$, the corresponding functions $x^i = x^i(z)$ is possible to determine from the system (6.11), the R. H. S. of which also confirms that $dx^i \neq F_i$.**

This conclusion is important since it shows that the two systems of coordinates should not be treated on an equal footing. This refers of course to the case of only one complex coordinate.

7 IS IT POSSIBLE TO HAVE A TWO COMPLEX COORDINATE DEPENDENCE OF THE GENERALIZED COORDINATES $X^i = X^i\left(\mathbf{x}(z)\,, z, v\right)$?

It will be proved below that such a case is impossible since it leads to an impossibility to determine the dependance X^i on the v coordinate.

Under the above assumption $X^i = X^i\left(\mathbf{x}(z)\,, z, v\right)$, the first set of three equations

$$dX^i = \frac{\partial X^i}{\partial x^1}dx^1 + \frac{\partial X^i}{\partial x^2}dx^2 + \frac{\partial X^i}{\partial x^3}dx^3 \tag{7.1}$$

can be represented as

$$F_i = \dot{X}^i \frac{\partial z}{\partial x^1}\dot{x}^1 dz + \dot{X}^i \frac{\partial z}{\partial x^2}\dot{x}^2 dz + \dot{X}^i \frac{\partial z}{\partial x^3}\dot{x}^3 dz = 3\dot{X}^i dz \quad , \tag{7.2}$$

so again relations (5.40) - (5.42) $\dot{X}^2 = \frac{F_2}{F_1}\dot{X}^1$, $\dot{X}^3 = \frac{F_3}{F_1}\dot{X}^1$ will hold.

The second set of equations

$$d^2X^i = dF_i(X, z) = dF_i(X(z, v), z) = dF_i(z, v) \tag{7.3}$$

will express the equality of the differentials, expressed in terms of the two different sets of coordinates (X, z) and (z, v)

$$\begin{aligned} d^2X^i &= \frac{\partial F_i}{\partial X^1}dX^1 + \frac{\partial F_i}{\partial X^2}dX^2 + \frac{\partial F_i}{\partial X^3}dX^3 + \frac{\partial F_i}{\partial z}dz = \\ &= \left[\frac{\partial F_i}{\partial X^1}\dot{X}^1 + \frac{\partial F_i}{\partial X^2}\dot{X}^2 + \frac{\partial F_i}{\partial X^3}\dot{X}^3 + \frac{\partial F_i}{\partial z}\right] dz + \\ &+ \left[\frac{\partial F_i}{\partial X^1}dX'^1 + \frac{\partial F_i}{\partial X^2}dX'^2 + \frac{\partial F_i}{\partial X^3}dX'^3\right] dv \quad . \end{aligned} \tag{7.4}$$

Taking into account that according to (5.42) $dX_1 = F_1$, $dX_2 = F_2$ and $dX_3 = F_3$ and also the expressed from (5.42) differential

$$dz = \frac{1}{3}\frac{F_1}{\dot{X}^1} \quad , \tag{7.5}$$

one can obtain for (7.4)

$$\begin{aligned} &\left[\frac{\partial F_i}{\partial X^1}dX'^1 + \frac{\partial F_i}{\partial X^2}dX'^2 + \frac{\partial F_i}{\partial X^3}dX'^3\right] dv = \\ &= \frac{2}{3}\left[\frac{\partial F_i}{\partial X^1}F_1 + \frac{\partial F_i}{\partial X^2}F_2 + \frac{\partial F_i}{\partial X^3}F_3\right] \quad . \end{aligned} \tag{7.6}$$

Dividing the L. H. S. and the R. H. S. for different values of the indice $i = 1, 2, 3$, one can obtain the following system of linear homogeneous algebraic equations in respect to X'^1, X'^2 and X'^3

$$\begin{aligned} &\left(\frac{\partial F_1}{\partial X^1}Q_2 - \frac{\partial F_2}{\partial X^1}Q_1\right) X'^1 + \left(\frac{\partial F_1}{\partial X^2}Q_2 - \frac{\partial F_2}{\partial X^2}Q_1\right) X'^2 + \\ &+ \left(\frac{\partial F_1}{\partial X^3}Q_2 - \frac{\partial F_2}{\partial X^3}Q_1\right) X'^3 = 0 \quad , \end{aligned} \tag{7.7}$$

$$\left(\frac{\partial F_1}{\partial X^1}Q_3 - \frac{\partial F_3}{\partial X^1}Q_1\right) X'^1 + \left(\frac{\partial F_1}{\partial X^2}Q_3 - \frac{\partial F_3}{\partial X^2}Q_1\right) X'^2 +$$

$$+\left(\frac{\partial F_1}{\partial X^3}Q_3-\frac{\partial F_3}{\partial X^3}Q_1\right)X'^3=0 \quad , \tag{7.8}$$

$$\left(\frac{\partial F_2}{\partial X^1}Q_3-\frac{\partial F_3}{\partial X^1}Q_2\right)X'^1+\left(\frac{\partial F_2}{\partial X^2}Q_3-\frac{\partial F_3}{\partial X^2}Q_2\right)X'^2+$$

$$+\left(\frac{\partial F_2}{\partial X^3}Q_3-\frac{\partial F_2}{\partial X^3}Q_2\right)X'^3=0 \quad , \tag{7.9}$$

where Q_i $(i=1,2,3)$ denotes the expression

$$Q_i\equiv\frac{\partial F_i}{\partial X_1}F_1+\frac{\partial F_i}{\partial X^2}F_2+\frac{\partial F_i}{\partial X^3}F_3 \quad . \tag{7.10}$$

Note that for the moment we have not yet used the equations $d^2X^i=dF_i=0$, from where $Q_i=0$. Then the system of equations (7.7) - (7.9) would be identically satisfied for all X'^1, X'^2 and X'^3 and it would be impossible to express them as solutions of the system. But even without making use of the equations $d^2X^i=dF_i=0$, the consistency (or inconsistensy) of the system (7.7) - (7.9) is a necessary condition for the consistency (or inconsistency) of the assumption about $X^i=X^i\left(\mathbf{x}(z),z,v\right)$.

Making use of the notation (6.1), the determinant of the system can be written as

$$\begin{vmatrix} \sum\limits_{l_1\neq 1}F_{l_1}\{F_1,F_2\}_{1,l_1} & \sum\limits_{l_2\neq 2}F_{l_2}\{F_1,F_2\}_{2,l_2} & \sum\limits_{l_3\neq 3}F_{l_3}\{F_1,F_3\}_{3,l_3} \\ \sum\limits_{m_1\neq 1}F_{m_1}\{F_1,F_3\}_{1,m_1} & \sum\limits_{m_2\neq 2}F_{m_2}\{F_1,F_3\}_{2,m_2} & \sum\limits_{m_3\neq 3}F_{m_3}\{F_1,F_3\}_{3,m_3} \\ \sum\limits_{n_1\neq 1}F_{n_1}\{F_2,F_3\}_{1,n_1} & \sum\limits_{n_2\neq 2}F_{n_2}\{F_2,F_3\}_{2,n_2} & \sum\limits_{n_3\neq 3}F_{n_3}\{F_2,F_3\}_{3,n_3} \end{vmatrix} \quad , \tag{7.11}$$

where instead of $\{F_i,F_j\}_{X^k,X^{n_k}}$ we have written only $\{F_i,F_j\}_{k,n_k}$.

The determinant is equal to

$$\sum_{l_k,m_k,n_k\ (l_k,m_k,n_k\neq k)} F_{l_1}F_{m_2}F_{n_3}\{F_1,F_2\}_{1,l_{l1}}\{F_1,F_3\}_{2,m_2}\{F_2,F_3\}_{3,n_3}-$$
$$-F_{l_1}F_{m_3}F_{n_2}\{F_1,F_2\}_{1,l_{l1}}\{F_1,F_3\}_{3,m_3}\{F_2,F_3\}_{2,n_2}-$$
$$-F_{l_1}F_{m1}F_{l_2}\{F_1,F_2\}_{1,m_{l1}}\{F_1,F_2\}_{2,l_2}\{F_2,F_3\}_{3,n_3}+$$
$$+F_{m_1}F_{l_3}F_{n_2}\{F_1,F_2\}_{1,m_{l1}}\{F_1,F_2\}_{3,l_3}\{F_2,F_3\}_{2,n_2}$$
$$+F_{n_1}F_{l_2}F_{m_3}\{F_2,F_3\}_{1,n_{l1}}\{F_1,F_2\}_{2,l_2}\{F_1,F_3\}_{3,m_3}-$$
$$-F_{n_1}F_{l_3}F_{n_2}\{F_2,F_3\}_{1,n_1}\{F_1,F_2\}_{3,l_3}\{F_1,F_3\}_{2,m_2} \quad . \tag{7.12}$$

It is seen from the above expression that the first and the third terms cancel but the other terms remain. Indeed, the explicite calculation of the determinant (7.11) gives the non-zero expression

$$\frac{\partial F_2}{\partial X^1}\{F_1,F_3\}_{2,3}+\frac{\partial F_3}{\partial X^1}\{F_1,F_2\}_{1,2} \quad . \tag{7.13}$$

Since the determinant is non-zero, the system of linear **homogeneous** algebraic equations does not have a solution and consequently the assumption that $X^i=X^i\left(\mathbf{x}(z),z,v\right)$ turns out to be **incorrect**.

8 COMPLEX STRUCTURE $X^i = X^i\left(\mathbf{x}(z,v)\,,z\right)$ OF THE GENERALIZED COORDINATES AND OF THE METRIC TENSOR COMPONENTS

Now it shall be proved that the parametrization (5.1) - (5.3) of the initially given cubic

algebraic curve (surface) can be extended to a parametrization in terms of a pair of complex coordinates (z, v) and thus a complex structure can be introduced. Of particular interest in view of possible physical applications to theories with extra dimensions and relation to ADS theories, which will be discussed in the Conlusion, will be the case of $v = \overline{z}$, when a pair of holomorphic - -antiholomorphic variables can be introduced.

In principle a manifold may admit a complex structure [44], if it can be covered with opened sets $U_{\ 1}, V_1, U_2, V_2$....., such that in any intersection $U_i \cap V_i$ the associated transformations $z^{k'} = z^{k'}(z_i, v_i)$ are complex (analytical) functions.

The investigated problem may be formulated as follows. Let (again) the system of equations (5.1) - (5.3) is given, subjected to the additional constraining equation $d^2X^i = 0$. **Then the parametrization (5.1) - (5.3) of the initially given cubic algebraic surface can be extended to a parametrization by means of a pair of complex coordinates (z, v) in the following way**

$$dX^i(\mathbf{X}) = F_i(\mathbf{X}(\mathbf{x}(z,v)), z) \quad . \tag{8.1}$$

Therefore, it should be proved that the same system of equations, investigated in the previous sections, is not contradictable under the assumption $X^i = X^i\left(\mathbf{x}(z,v)\,,z\right)$.

The first set of equations to be used is similar to (7.3), but this time expressing the equality of the differentials

$$dF_i(\mathbf{X}(z,v), z) \ = dF_i(\mathbf{x}(z,v), z) \qquad , \tag{8.2}$$

written in terms of the coordinates ($\mathbf{X}$,z) and ($\mathbf{x}$, z)

$$\begin{aligned}
&\left[\frac{\partial F_i}{\partial X^1}\dot{X}^1 + \frac{\partial F_i}{\partial X^2}\dot{X}^2 + \frac{\partial F_i}{\partial X^3}\dot{X}^3 + \frac{\partial F_i}{\partial z}\right] dz + \\
&+ \left[\frac{\partial F_i}{\partial X^1}X^{\prime 1} + \frac{\partial F_i}{\partial X^2}X^{\prime 2} + \frac{\partial F_i}{\partial X^3}X^{\prime 3}\right] dv = \\
&= \left[\frac{\partial F_i}{\partial x^1}\dot{x}^1 + \frac{\partial F_i}{\partial x^2}\dot{x}^2 + \frac{\partial F_i}{\partial x^3}\dot{x}^3 + \frac{\partial F_i}{\partial z}\right] dz + \\
&+ \left[\frac{\partial F_i}{\partial x^1}x^{\prime 1} + \frac{\partial F_i}{\partial x^2}x^{\prime 2} + \frac{\partial F_i}{\partial x^3}x^{\prime 3}\right] dv \qquad .
\end{aligned} \tag{8.3}$$

The second set of equations takes into account the fact that the second differential d^2X^i is zero, or equivalently

$$d^2X^i = dF_i(\mathbf{X}(z,v), z) = 0 \qquad , \tag{8.4}$$

where $dF_i(\mathbf{X}(z,v),z)$ is given by the L. H. S. of equation (8.3).

The third set of equations is

$$dX^i = F^i = \frac{\partial X_i}{\partial z}dz + \frac{\partial X_i}{\partial v}dv \quad . \tag{8.5}$$

Let us now introduce the notations

$$M_i(X,z) \equiv \frac{\partial F_i}{\partial X^k}\dot{X}^k \quad ; \qquad M_i(x,z) \equiv \frac{\partial F_i}{\partial x^k}\dot{x}^k \quad , \tag{8.6}$$

$$M_i(X,v) \equiv \frac{\partial F_i}{\partial X^k}X^{\prime k} \quad ; \qquad M_i(x,v) \equiv \frac{\partial F_i}{\partial x^k}x^{\prime k} \quad , \tag{8.7}$$

which will allow us to write down the the first and the second set of equations (7.3) - (7.4) in the following compact form

$$\left[M_i(X,z) - M_i(x,z)\right]dz + \left[M_i(X,v) - M_i(x,v)\right]dv = 0 \quad , \tag{8.8}$$

$$\left[M_i(X,z) + \frac{\partial F_i}{\partial z}\right]dz + M_i(X,v)dv = 0 \quad . \tag{8.9}$$

Expressing $\frac{\partial X^i}{\partial v}dv$ from (8.5), it can easily be proved that

$$M_i(X,v)dv = \frac{\partial F_i}{\partial X^k}X^{\prime k}dv = -\frac{\partial F_i}{\partial X^k}\dot{X}^k dz + \frac{\partial F_i}{\partial X^k}F_k \quad , \tag{8.10}$$

where the last term is zero due to the fulfillment of the second set of equations (8.5). Consequently, from (8.10) it follows

$$M_i(X,z)dz + M_i(X,v)dv = dF_i(z,v) = dF_i(\mathbf{X}(z,v),z) = 0 \quad . \tag{8.11}$$

Additionally, if (8.11) is substracted from (8.8) and (8.9), one easily obtains

$$M_i(x,z)dz + M_i(x,v)dv = dF_i(z,v) = dF_i(\mathbf{x}(z,v),z) = 0 \quad , \tag{8.12}$$

$$\frac{\partial F_i}{\partial z}dz = 0 \quad \Rightarrow \quad \frac{\partial F_i}{\partial v}dv = 0 \quad . \tag{8.13}$$

In other words, if the differential $dF_i(\mathbf{X}(z,v),z)$ is zero in terms of the coordinates ($\mathbf{X}$,z), then it necessarily should be zero in the coordinates $(\mathbf{x},z)$. But in the spirit of the discussion at the end of section 6, this does not mean that if $dX^i = F_i$, then the same should hold also for the initial coordinates x^i, i.e. $dx^i \neq F_i$. Indeed, we can find

$$M_i(x,z) \equiv \frac{\partial F_i}{\partial x^k}\dot{x}^k = \frac{\partial F_i}{\partial X^l}\frac{\partial X^l}{\partial x^k}\dot{x}^k =$$

$$= \frac{\partial F_i}{\partial X^l}\left[\frac{\dot{X}^l}{\dot{x}^k} + \frac{X^{\prime l}}{x^{\prime k}}\right]\dot{x}^k = M_i(X,z) + M_i(X,v)\frac{\dot{x}^k}{x^{\prime k}} \quad . \tag{8.14}$$

Similarly

$$M_i(x,v) = M_i(X,v) + M_i(X,z)\frac{x'^k}{\dot{x}^k} \quad . \tag{8.15}$$

If the above two expressions are substituted into (8.12), and (8.11) is taken into account, one can obtain

$$M_i(X,v)\frac{\dot{x}^k}{x'^k}dz + M_i(X,z)\frac{x'^k}{\dot{x}^k}dv = 0 \quad . \tag{8.16}$$

Additionally, we have

$$M_i(X,v) = M_i(X,z)\frac{x'^k}{\dot{x}^k} \quad ; \qquad dv = -\frac{M_i(x,z)}{M_i(x,v)}dz \quad . \tag{8.17}$$

Therefore, the following equation in partial derivatives in respect to $F_i = F_i(x^l)$ can be derived

$$\frac{\partial F_i}{\partial x^l}x'^l\frac{\dot{x}^k}{x'^k}\frac{x'^m}{\dot{x}^m} - \frac{\partial F_i}{\partial x^l}\dot{x}^l\frac{x'^k}{\dot{x}^k} = 0 \quad . \tag{8.18}$$

As partial case, for an arbitrary $\frac{\partial F_i}{\partial x^l}$ ($l = l_1$ is fixed; there is an independent summation along the indices k and m), the above equation holds when the following complicated nonlinear equation in partial derivatives is satisfied

$$x'^{l_1}\frac{\dot{x}^k}{x'^k}\frac{x'^m}{\dot{x}^m} - \dot{x}^{l_1}\frac{x'^k}{\dot{x}^k} = 0 \quad . \tag{8.19}$$

Note that the equation takes a different form if one assumes that

$$dx^i = F_i \qquad \Rightarrow \quad dv = \frac{F_i}{x'^i} - \frac{\dot{x}^i}{x'^i}dz \quad . \tag{8.20}$$

Then instead of (8.19), the obtained equation will be of the form

$$\frac{\dot{x}^k}{x'^k}\left(\frac{x'^m}{\dot{x}^m}\right)^2 - \frac{x'^k}{\dot{x}^k} = 0 \quad . \tag{8.21}$$

In fact, even a stronger statement may be proved, clearly showing that from $dF_i(X,z) = dF_i(x,z) = 0$ and $dX_i = F_i$ it does not follow that $dx^i = F_i$. If expression (8.17) for $M_i(X,v) = M_i(X,z)\frac{x'^k}{\dot{x}^k}$ is substituted into (8.11), one obtains

$$M_i(X,z)\left[dz + \frac{x'^k}{\dot{x}^k}dv\right] = 0 \tag{8.22}$$

and since $M_i(X,z) \neq 0$ (and if $\frac{\partial F_i}{\partial X^l} \neq 0$), it follows

$$dx^k = \dot{x}^k dz + x'^k dv = 0 \quad . \tag{8.23}$$

9 ANALYSIS OF THE FOURTH AND THE FIFTH SET OF EQUATIONS FOR THE PREVIOUS CASE $X^i = X^i(\mathbf{x}(z,v), z)$

The **fourth set** of equations, which will be considered is

$$dX^k = \frac{\partial X^k}{\partial x^1}dx^1 + \frac{\partial X^k}{\partial x^2}dx^2 + \frac{\partial X^k}{\partial x^3}dx^3 =$$

$$= 3\dot{X}^k dz + X'^k \frac{\dot{x}^m}{x'^m}dz + \dot{X}^k \frac{x'^m}{\dot{x}^m}dv + X'^k dv \quad . \tag{9.1}$$

If multiplied by $\frac{\partial F_i}{\partial X^k}dzdv$ and also relation (8.11) $M_i(X,z)dz + M_i(X,v)dv = 0$ is taken into account, the fourth set of equations can be written as

$$\frac{\partial F_i}{\partial X^k}F_k dv = M_i(X,z)dz\left[\frac{x'^m}{\dot{x}^m}(dv)^2 - \frac{\dot{x}^m}{x'^m}(dz)^2\right] \quad . \tag{9.2}$$

The **fifth set** of equations is

$$d^2X^k = 0 = \frac{\partial^2 X^k}{\partial x^m \partial x^n}dx^m dx^n + \frac{\partial X^k}{\partial x^m}d^2x^m \quad , \tag{9.3}$$

where

$$\frac{\partial^2 X^k}{\partial x^m \partial x^n} = \frac{1}{2}\left[\frac{\partial}{\partial z}\left(\frac{\partial X^k}{\partial x^m}\right)\frac{\partial z}{\partial x^n} + \frac{\partial}{\partial v}\left(\frac{\partial X^k}{\partial x^m}\right)\frac{\partial v}{\partial x^n}\right] = \tag{9.4}$$

$$= \frac{1}{2}\{\frac{\ddot{X}^k}{\dot{x}^m \dot{x}^n} + \frac{X''^k}{x'^m x'^n} + \dot{X}'^k\left[\frac{1}{\dot{x}^m x'^n} + \frac{1}{\dot{x}^n x'^m}\right] -$$

$$-\dot{X}^k\left[\frac{\dot{x}'^m}{x'^n} + \frac{\ddot{x}^m}{\dot{x}^n}\right] - \frac{X'^k}{(x'^m)^2}\left[\frac{\dot{x}'^m}{\dot{x}^n} + \frac{x''^m}{x'^n}\right]\} \quad , \tag{9.5}$$

$$\frac{\partial X^k}{\partial x^m} = \frac{\dot{X}^k}{\dot{x}^m} + \frac{X'^k}{x'^m} \quad , \tag{9.6}$$

$$dx^m dx^n = \dot{x}^m \dot{x}^n (dz)^2 + (x'^m \dot{x}^n + \dot{x}^m x'^n)dzdv + x'^m x'^n (dv)^2 \quad , \tag{9.7}$$

$$d^2x^m = \ddot{x}^m (dz)^2 + x''^m (dv)^2 + 2\dot{x}'^m dxdv + \dot{x}^m d^2z + x'^m d^2v \quad . \tag{9.8}$$

Our goal further will be to show whether the fifthe equation (9.3) constitutes a separate equation or whethetr it follows from the preceeding four ones.

For the purpose, let us multiply both sides of the fifth equation by $\frac{\partial F_i}{\partial X^k}$ and see which are the terms, containing the second differentials d^2z and d^2v

$$\left(\frac{\partial F_i}{\partial X^k}\frac{\dot{X}^k}{\dot{x}^m} + \frac{\partial F_i}{\partial X^k}\frac{X'^k}{x'^m}\right)\left(\dot{x}^m d^2z + x'^m d^2v\right) =$$

$$= M_i(x,z)d^2z + M_i(x,v)d^2v \quad . \tag{9.9}$$

In (9.9) we have used relations (8.14) and (8.15) for $M_i(x,z)$ and $M_i(x,v)$. But we may note that the obtained term in (9.9) can be found from the relation (8.12) $M_i(x,z)dz + M_i(x,v)dv = 0$, if it is differentiated by z and v and the resulting equations are summed up. Therefore

$$M_i(x,z)d^2z + M_i(x,v)d^2v = -M_i(x,z)(dz)^2 - M_i(x,v)(dv)^2 -$$

$$-(\dot{M}_i(x,v) + M_i^{'}(x,z))dxdv \quad . \tag{9.10}$$

The derivatives $\dot{M}_i(x,v)$ and $M_i^{'}(x,z)$ can be found also from the already used expressions (8.14) and (8.15)

$$M_i^{'}(x,z) = M_i^{'}(X,z) + \frac{\dot{x}^m}{x^{'m}} M_i^{'}(X,v) +$$

$$+ M_i(X,v)\frac{\left[\dot{x}^{'m}x^{'m} - \dot{x}^m x^{''m}\right]}{\left(x^{'m}\right)^2} \quad , \tag{9.11}$$

$$\dot{M}_i(x,v) = \dot{M}_i(X,v) + \frac{x^{'m}}{\dot{x}^m}\dot{M}_i(X,z) +$$

$$+ M_i(X,z)\frac{\left[\dot{x}^{'m}\dot{x}^m - x^{'m}\dot{x}^m\right]}{\left(x^{'m}\right)^2} \quad . \tag{9.12}$$

Making use of all the expressions (9.9) - (9.12), the following expression for the fifth equation (9.3), multiplied by $\frac{\partial F_i}{\partial X^k}$, can be obtained:

$$(dz)^2[-2M_i(X,z) + M_i(X,z)\frac{\ddot{x}^m}{\dot{x}^m} + \frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}\dot{x}^m\dot{x}^n] +$$

$$+(dv)^2[-2\frac{x^{'m}}{\dot{x}^m}M_i(X,z) + M_i(X,z)\frac{x^{''m}}{\dot{x}^m} + \frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}x^{'m}x^{'n}] +$$

$$+dzdv[2M_i(X,z)\frac{\dot{x}^{'m}}{\dot{x}^m} + 2\frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}\dot{x}^m x^{'n} - 2M_i^{'}(X,z) -$$

$$-2\dot{M}_i(X,z)\frac{x^{'m}}{\dot{x}^m} + 2M_i(X,z)\frac{\left[x^{'m}\ddot{x}^m - \dot{x}^{'m}\dot{x}^m\right]}{\left(\dot{x}^m\right)^2}] = 0 \quad . \tag{9.13}$$

The last two terms $\frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}\dot{x}^m\dot{x}^n$ and $\frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}x^{'m}x^{'n}$ in the first two square brackets can be found as follows: First, the derivatives $\dot{M}_i(X,v)$ and $M_i^{'}(X,v)$ can be expressed from the relation (8.17) $M_i(X,v) = M_i(X,z)\frac{x^{'k}}{\dot{x}^k}$:

$$\dot{M}_i(X,v) = \dot{M}_i(X,z)\frac{x^{'m}}{\dot{x}^m} + M_i(X,z)\frac{\left[\dot{x}^{'m}\dot{x}^m - x^{'m}\ddot{x}^m\right]}{\left(\dot{x}^m\right)^2} \quad , \tag{9.14}$$

$$M_i^{'}(X,v) = M_i^{'}(X,z)\frac{x'^m}{\dot{x}^m} + M_i(X,z)\frac{\left[x''^m \dot{x}^m - x'^m \dot{x}'^m\right]}{\left(\dot{x}^m\right)^2} \quad . \tag{9.15}$$

But on the other hand, the same derivatives can be found by using the defining expressions (8.6 - 8.7)

$$\dot{M}_i(X,v) = \frac{\partial^2 F_i}{\partial X^k \partial X^l}\frac{\partial X^l}{\partial x^m}\frac{\partial X^k}{\partial x^n}\dot{x}^m x'^n + \frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}\dot{x}^n x'^m +$$

$$+ M_i(X,z)\frac{x'^m \dot{x}^n}{\dot{x}^m} + M_i(X,z)\dot{x}'^n \quad , \tag{9.16}$$

$$M_i^{'}(X,v) = \frac{\partial^2 F_i}{\partial X^k \partial X^l}\frac{\partial X^l}{\partial x^m}\frac{\partial X^k}{\partial x^n}x'^m x'^n + \frac{\partial F_i}{\partial X^k}\frac{\partial^2 X^k}{\partial x^m \partial x^n}x'^n x'^m +$$

$$+ \frac{\partial F_i}{\partial X^k}\frac{\partial X^k}{\partial x^n}\frac{x'^m \dot{x}'^n}{\dot{x}^m} + \frac{\partial F_i}{\partial X^k}\frac{\partial X^k}{\partial x^m}x''^n \quad . \tag{9.17}$$

Therefore, the desired expressions can be found by setting up formulae (9.14) equal to (9.15) and also formulae (9.15) equal to (9.17). It can easily be derived how eq. (9.13) will transform, but unfortunately, this would not result in any simplification of the equation in respect to the generalized coordinates X^i.

It remained only to show whether the fourth equation (8.2) is independent from the preceeding ones and thus can be treated separately or if it follows naturally from these equations. For the purpose, let us take the differential of (8.2) and use equations (8.10). After some lengthy, but straightforward calculations it can be obtained

$$(dz)(dv)^2[\dot{M}_i(X,z)\frac{x'^m}{\dot{x}^m} + 2\frac{x'^m}{\dot{x}^m}M_i^{'}(X,z) - M_i(X,z)\frac{\left(\dot{x}'^m \dot{x}^m - x'^m \ddot{x}^m\right)}{\left(\dot{x}^m\right)^2} +$$

$$+ M_i(X,z)\frac{2\left(\dot{x}'^m \dot{x}^m - x'^m \ddot{x}^m + x''^m \dot{x}^m - \dot{\dot{x}}'^m x'^m\right)}{\left(\dot{x}^m\right)^2}] +$$

$$+ (dz)^2(dv)[M_i^{'}(X,z)\frac{\dot{x}^m}{x'^m} + 2\dot{M}_i(X,z) + M_i(X,z)\frac{\left(\dot{x}'^m x'^m - \dot{x}^m x''^m\right)}{\left(x'^m\right)^2}] +$$

$$+ (dz)^3[\dot{M}_i(X,z)\frac{\dot{x}^m}{x'^m} + M_i(X,z)\frac{\left(\ddot{x}^m x'^m - \dot{x}^m \dot{x}'^m\right)}{\left(x'^m\right)^2}] +$$

$$+ (dv)^3[M_i^{'}(X,z)\frac{x'^m}{\dot{x}^m} + M_i(X,z)\frac{\left(x''^m \dot{x}^m - x'^m \dot{x}'^m\right)}{\left(\dot{x}^m\right)^2}] = 0 \quad . \tag{9.18}$$

This is an equation both for the initial coordinates x^i and for the generalized ones X^i. The equation would have been only for the initial coordinates if the sum of all the terms

with $\dot{M}_i(X,z)$ and $M_i^{'}(X,z)$ is sero. However, it can be found by using relation (8.23) $dx^m = 0$, that the sum of these terms is equal to

$$-2M_i^{'}(X,z)\left[dz(dv)^2 + (dz)^2 dv\right] =$$

$$= -2M_i^{'}(X,z)(dz)^3\left[\left(\frac{\dot{x}^m}{x^{'m}}\right)^2 - \frac{\dot{x}^m}{x^{'m}}\right] \quad . \tag{9.19}$$

Also, evidently the fourth equation (8.18) is different from the fifth one (9.13).

It can easily be seen that if the initial coordinates are known, equation (9.18) can be treated as an differential equation in respect to $M_i(X,z)$.

10 DISCUSSION - OBTAINED RESULTS AND POSSIBLE APPLICATIONS IN THEORIES WITH EXTRA DIMENSIONS

10.1. SUMMARY OF THE OBTAINED RESULTS

In this paper we continued the investigation of cubic algebraic equations in gravity theory, which has been initiated in the previous paper [10].

It has been demonstrated that there is a wide variety of algebraic equations of third, fourth, fifth, ninth and tenth order. Their derivation is based on two important initial assumptions:

1. The covariant and contravariant metric components are treated independently, which is a natural approach in the framework of affine geometry [15 - 18].

2. Under the above assumption, the gravitational Lagrangian (or Ricci tensor) should remain the same as in the standard gravitational theory with inverse contravariant metric tensor components.

It has been proved in Appendix A that if the contravariant metric tensor components are assumed to be represented as a factorized product of the components of the vector field $dX = (dX^1, dX^2, ..., dX^n)$, lying in the tangent space of the given manifold, i.e. $\widetilde{g}^{ij} = dX^i dX^j$, then the new connection $\widetilde{\Gamma}_{ij}^k = \frac{1}{2} dX^k dX^s(g_{js,i} + g_{is,j} - g_{ij,s})$ has again an affine connection transformation property.

The proposed approach allows to treat the Einstein's equations as algebraic equations, and **thus to search for separate classes of solutions for the covariant and contravariant metric tensor components.** The mathematical approach, based again on the application of the linear-fractional transformation, will be developed in the third part of this paper. It can be supposed also that the existence of such separate classes of solutions might have some interesting and unexplored until now physical consequences.

Some of them will be demonstrated in reference to theories with extra dimensions, but no doubt the physical applications are much more numerous.

Since the derived solutions should be of a most general type (for example - non-zero covariant derivative of the metric tensor), the "transition" to the standard Einsteinian theory of gravity can be performed by investigating the intersection with the corresponding algebraic equations. For example, the zero covariant derivative of the metric tensor represents again a cubic algebraic equation, while the condition for the existence of an contravariant inverse metric tensor can be considered as a quadratic algebraic equation $g_{ij}dX^{j}dX^{k}=\delta_{i}^{k}$ in respect to the tangent space vector components dX^{i}. On the contrary, if dX^{i} are considered to be known, then the covariant components g_{ij} satisfy a **system of linear operator algebraic equations** (since the unknown variables g_{ij} form a matrix and not a vector-column, as in the usual case of linear algebraic systems). This system is well defined irrespectively of the fact that det $\parallel dX^{i}dX^{j}\parallel=0$ and moreover, in Appendix C it has been shown how such a system **in the general $N-$dimensional case can be transformed to a system of linear algebraic equations.** For that purpose, an original approach has been consistently developed for the first time, **based on the block structure method.** In a purely algebraic aspect, further investigation is needed to establish how the type of solutions of the predetermined linear algebraic system is related to the matrix $dX^{i}dX^{j}$, characterizing the initially given operator system of equations.

These are more or less the "conceptual" issues (excluding of course the developed method in respect to the system of linear operator equations) for further investigation, raised in this first part of the present paper.

The most important result in this paper is related to the possibility to find the uniformization functions for a multicomponent cubic algebraic surface. More concretely, the derived in [10] s.c. "cubic algebraic equation (2.14) of reparametrizational invariance of the gravitational Lagrangian" is investigated by performing consequently the linear - fractional transformations (3.5) and (3.24) in respect to the variables dX^{3} and dX^{2}. It is important to stress that (2.14) is a **multi - variable cubic algebraic equation** (i. e. algebraic surface), which for the three - dimensional case can be represented as

$$2p\Gamma_{11}^{r}g_{1r}(dX^{1})^{3}+2p\Gamma_{22}^{r}g_{2r}(dX^{2})^{3}+2p\Gamma_{33}^{r}g_{3r}(dX^{3})^{3}+$$
$$+6p\Gamma_{13}^{r}g_{3r}dX^{1}(dX^{3})^{2}+6p\Gamma_{23}^{r}g_{3r}(dX^{2})(dX^{3})^{2}+2p(\Gamma_{22}^{r}g_{3r}+$$
$$+2\Gamma_{32}^{r}g_{2r})(dX^{2})^{2}(dX^{3})+2p(\Gamma_{11}^{r}g_{3r}+2\Gamma_{13}^{r}g_{1r})(dX^{1})^{2}(dX^{3})+$$
$$+2p(2\Gamma_{12}^{r}g_{2r}+\Gamma_{22}^{r}g_{1r})(dX^{2})^{2}(dX^{1})+2p(\Gamma_{11}^{r}g_{2r}+$$
$$+2\Gamma_{12}^{r}g_{1r})(dX^{1})^{2}(dX^{2})+4p(\Gamma_{12}^{r}g_{3r}+\Gamma_{3(1}^{r}g_{2)r})dX^{1}dX^{2}dX^{3}-$$
$$-2R_{12}dX^{1}dX^{2}-2R_{13}dX^{1}dX^{3}-2R_{23}dX^{2}dX^{3}-$$
$$-R_{11}(dX^{1})^{2}-R_{22}(dX^{2})^{2}-R_{33}(dX^{3})^{2}=0 \qquad (10.1)$$

At the same time, all we know from standard algebraic geometry [9] is how to parametrize a plane (i. e. a two - dimensional) cubic algebraic equation of the form (3.14) $\widetilde{n}^{2}=$

$4m^3 - g_2 m - g_3$. Therefore, the advantage of applying the linear- fractional transformations (3.5) and (3.23) is that by adjusting their coefficient functions (so that the highest - third degree in the transformation equation will vanish), the following sequence of plane cubic algebraic equations is fulfilled (the analogue of eq.(65) in [10]):

$$P_1^{(3)}(n_{(3)})m_{(3)}^3 + P_2^{(3)}(n_{(3)})m_{(3)}^2 + P_3^{(3)}(n_{(3)})m_{(3)} + P_{(4)}^{(3)} = 0 \qquad , \qquad (10.2)$$

$$P_1^{(2)}(n_{(2)})m_{(2)}^3 + P_2^{(2)}(n_{(2)})m_{(2)}^2 + P_3^{(2)}(n_{(2)})m_{(2)} + P_{(4)}^{(2)} = 0 \qquad , \qquad (10.3)$$

$$P_1^{(1)}(n_{(1)})m_{(1)}^3 + P_2^{(1)}(n_{(1)})m_{(1)}^2 + P_3^{(1)}(n_{(1)})m_{(1)} + P_{(4)}^{(1)} = 0 \qquad , \qquad (10.4)$$

where $m_{(3)}$, $m_{(2)}$, $m_{(1)}$ denote the ratios $\frac{a_3}{c_3}$, $\frac{a_2}{c_2}$, $\frac{a_1}{c_1}$ in the corresponding linear - fractional transformations and $n_{(3)}$, $n_{(2)}$, $n_{(1)}$ are the "new" variables $\widetilde{dX}^3$, $\widetilde{dX}^2$, $\widetilde{dX}^1$. The sequence of plane cubic algebraic equations (10.2 - 10.4) should be understood as follows: the first one (10.2) holds if the second one (10.3) is fulfilled; the second one (10.3) holds if the third one (10.4) is fulfilled. Of course, in the case of n variables (i. e. n component cubic algebraic equation) the generalization is straightforward. Further, since each one of the above plane cubic curves can be transformed to the algebraic equation ($i = 1, 2, 3$)

$$\widetilde{n}_{(i)}^2 = \overline{P}_1^{(i)}(\widetilde{n}_{(i)})m_{(i)}^3 + \overline{P}_2^{(i)}(\widetilde{n}_{(i)})m_{(i)}^2 + \overline{P}_3^{(i)}(\widetilde{n}_{(i)})m_{(i)} + \overline{P}_4^{(i)}(\widetilde{n}_{(i)}) \qquad (10.5)$$

and subsequently to its parametrizable form, one obtains the solutions (5.1 - 5.3) of the initial multicomponent cubic algebraic equation. **Consequently, these expressions are nothing else but the uniformization functions for the variables dX^1, dX^2, dX^3 of the cubic equation.** This is so because in Sections 5 - 9 it has been shown that from the expressions (5.1 - 5.3) a system of first - order nonlinear differential equations is obtained, for which always a solution $X^1 = X^1(z)$, $X^1 = X^1(z)$, $X^1 = X^1(z)$ exists. Thus the dependence on the generalized coordinates X^1, X^2, X^3 in the uniformization functions (5.1 - 5.3) dissappears and only the dependence on the complex coordinate z remains, as it should be for uniformization functions.

Moreover, the initial assumption $dX^i = 0$ for obtaining the solutions (5.1 - 5.3) allows us to derive a system of nonlinear differential equations also for the initial variables x^1, x^2, x^3 and thus the corresponding solutions $x^1 = x^1(z)$, $x^2 = x^2(z)$, $x^3 = x^3(z)$ in principle can be found. This analysis has been performed in section 6. In fact, it can easily be guessed that if we have the solutions $X^1 = X^1(z)$, $X^2 = X^2(z)$, $X^3 = X^3(z)$ and the additional condition $d^2X^i = 0$ (which in fact relates the generalized and the initial sets of coordinates), then the solutions $x^1 = x^1(z)$, $x^2 = x^2(z)$, $x^3 = x^3(z)$ should also be "coordinated" with the previous ones. Indeed, this is evident from the dependence of the functions $\overline{S}_1^{(i)}$,$\overline{S}_2^{(i)}$,....,$\overline{S}_6^{(i)}$ in the system (6.11) for $\frac{dx^i}{dz}$ both on the functions $\frac{\partial F_i}{\partial X^k}$ and $\frac{\partial F_i}{\partial x^k}$, i.e. on both system of coordinates. Of particular importance is the conclusion at the end of Sect. VI that the two sets of coordinates X^1, X^2, X^3 and x^1, x^2, x^3 should not be treated on an equal footing. This means that if dX^1, dX^2, dX^3 satisfy the originally derived cubic algebraic equation, then it is not necessary to assume this for dx^1, dx^2, dx^3.

Much more interesting is the other investigated case in Sections 7 - 9, where a pair of complex coordinates z, v has been introduced and thus through the generalized coordinates $X^1 = X^1(z, v)$, $X^2 = X^2(z, v)$, $X^3 = X^3(z, v)$ a complex structure of the metric tensor components is introduced. For the investigated case under the assumption $d^2 X^i = 0$ there is only one way for introducing this complex structure - namely, through the dependence of the initial coordinates on z and v, i. e. $X^i = X^i(\mathbf{x}(z, v), z)$. Otherwise, if some other possibility is assumed, for example $X^i = X^i(\mathbf{x}(z), z, v)$, then, as proved in section 7, the obtained system of equations is contradictory. Therefore, it remains to investigate the full system of equations for the only allowed case $X^i = X^i(\mathbf{x}(z, v), z)$, which has been performed in Sections 8 and 9. Remarkably, a nonlinear differential equation is obtained only for the initial coordinates. However, no such an equation only for the generalized coordinates can be obtained - the derived equation depends in a complicated manner on both system of coordinates. Since the existence of these noncontradictory systems of equations confirms that a complex structure can be introduced, one may express the line element $ds^2 = g_{ij}(\mathbf{X}) dX^i dX^j$ as

$$ds^2 = \widetilde{g}_{zz}(z, v)(dz)^2 + \widetilde{g}_{zv}(z, v) dz dv + \widetilde{g}_{vv}(z, v)(dv)^2 \quad , \tag{10.6}$$

where

$$\widetilde{g}_{zz}(z, v) \equiv g_{ij}(\mathbf{X}(z, v)) \dot{X}^i \dot{X}^j \quad ; \quad \widetilde{g}_{vv}(z, v) \equiv g_{ij}(\mathbf{X}(z, v)) X'^i X'^j \quad , \tag{10.7}$$

$$\widetilde{g}_{zv}(z, v) \equiv g_{ij}(\mathbf{X}(z, v)) \left[\dot{X}^i X'^j + X'^i \dot{X}^j \right] \quad . \tag{10.8}$$

This result will be of particular importance in reference to possible physical applications.

10.2. SOME POSSIBLE PHYSICAL APPLICATIONS IN GRAVITATIONAL THEORIES WITH EXTRA DIMENSIONS

10.2.1. FUNDAMENTAL PARALELLOGRAM ON THE COMPLEX PLANE, ORBIFOLD COMPACTIFICATION AND PERIODIC IDENTIFICATION

As it is well - known, a class of two - dimensional metrics exists [45]

$$ds^2 = R^2 \frac{(a^2 - v^2) du^2 + 2uv du dv + (a^2 - u^2) dv^2}{(a^2 - u^2 - v^2)^2} \quad , \tag{10.9}$$

representing the linear element of a unit surface in the Lobachevsky space with a constant negative curvature $-\frac{1}{R^2}$. Performing the transformations

$$\frac{a^2 - u\, u_0 - v v_0}{\sqrt{a^2 - u^2 - v^2}} = a e^{-\frac{\rho}{R}} \quad ; \quad \frac{u_0 v - u v_0}{a^2 - u\, u_0 - v v_0} = \frac{\sigma}{R} \quad , \tag{10.10}$$

the above metric (10.9) can be rewritten as

$$ds^2 = d\rho^2 + e^{-\frac{2\rho}{R}} d\sigma^2 \qquad , \tag{10.11}$$

which turns out to be the same as the metric

$$ds^2 = e^{-2kr_{-}\Phi}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 d\Phi^2 \qquad , \tag{10.12}$$

extensively used in the first version of the Randall - Sundrum model [[46]. In (10.12) $\eta_{\mu\nu}$ is the flat Minkowski metric, $0 \leq \Phi \leq \pi$ and the extra dimension is a finite interval, whose size is set by the compactification radius r_c. A nice and effective generalization of this model implies that the SM (Standard Model) particles and forces with the exception of gravity are confined to a 4 - dimensional subspace, but within a $(4+n)$- dimensional spacetime.

The first problem in reference to Lobachevsky geometries and theories with extra dimensions is the following one: **starting from the $5-$dimensional metric (10.12) in its relatively simple form and applying the developed in this paper approach, to find its equivalent form in the two-dimensional coordinates (z, v).** Obviously, a new class of Lobachevsky metric, depending on the Weierstrass function and its derivative can be obtained. Remember also that the complex coordinate z is related to the Weierstrass function and as mentioned, it is defined on the lattice $\Lambda = \{m\omega_1 + n\omega_2 \mid m, n \in Z;$ $\omega_1, \omega_2 \in C, Im\frac{\omega_1}{\omega_2} > 0\}$ on the two - dimensional projective plane CP^2. Then let us define the complex uniformization coordinate z as $z = \pi r_c(\cos\Phi + i\sin\Phi)$ and $0 \leq \Phi \leq \pi$ is the periodic coordinate. Under the transformation $\Phi = arctg\frac{z}{r_c}$, the metric (10.12) will transform as

$$ds^2 = e^{-2kr_{-}\frac{r_c}{\sqrt{z^2+r_c^2}}}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{r_c^4}{\sqrt{z^2+r_c^2}}dz^2 \qquad . \tag{10.13}$$

Now the advantage of such a formulation is clear: the nice properties of the Weierstrass function and its derivative

$$\rho^{'}(z + \omega_i) = \rho^{'}(z) \quad ; \quad \rho(\pi r_c) = \rho(-\pi r_c) \tag{10.14}$$

exactly matches the requirement for orbifold identification of the points $+\pi r_c$ and $-\pi r_c$. **In other words, by making the above transformation for the periodical coordinate Φ of the additional extra dimension [46, 47], a periodical identification is achieved of the identical points under orbifold compactification with a fundamental domain of lenght $2\pi r_c$ with the lattice points of the fundamental paralellogram on the complex plane**. A general overview of orbifold compactifications is presented in [48]. However, one should not expect also a periodic identification $X_\mu \rightarrow X_\mu + 2\pi r_c$ for the generalized coordinates of the flat Minkowski space, since there is no strictly proved theorem that the solutions $X^1 = X^1(z, v)$, $X^2 = X^2(z, v)$, $X^3 = X^3(z, v)$ and $X^4 = X^4(z, v)$ of the system of equations (3.11), (3.22) and (3.29) (more exactly, the equivalent to it system, written for the five - dimensional case of the metric (10.12)) should depend on z only through the Weierstrass function and its derivative.

10.2.2. FACTORIZATION AND NON - FACTORIZATION OF THE VOLUME ELEMENT

Obtaining some estimates for the fundamental lenght $2\pi r_c$ would be interesting, since for d additional compactified dimensions, each one of radius r_i, the fundamental (Planck) scale of gravity is related to the gravity scale in the $(4+d)$- dimensional space as [49, 50, 51]

$$M_{pl}^2 = M_{fund}^{2+d} r_i^d = M_{fund}^{2+d} \, V^d \quad . \tag{10.15}$$

Obtaining an estimate of the volume of the extradimensiona space is important, since by taking a large volume the large discrepancy between the Planck scale of 10^{19}GeV and

the electroweak scale of 100 GeV can be diminished and thus the hierarchy problem can be solved. For a derivation of the relation (10. 15) between the gravity scales on the base of dimensional analysis of the higher - dimensional Einstein - Hilbert action, one may use the excellent review article [52]. In such a case, the metric (10.1) should be generalized to the $(d+4)$- dimensional metric of the Lobachevsky space. Naturally, the most simple case [52] is of a flat extradimensional space, when $V^d = (2\pi r\ )^d$ and also a flat $4D$ Minkowski metric. However, it would be much more interesting to consider an $(d+4)$ - dimensional ADS (Lobachevsky) space with a constant negative curvature, whose volume element is given by [53]

$$dV_n = \frac{c_n dx_1 dx_2 ... dx_n}{(c^2 - x_\alpha x_\alpha)^{\frac{(n+1)}{2}}} \tag{10.16}$$

and can be found by splitting up the n- dimensional volume by means of $(n-1)$- dimensional hyperplanes, perpendicular to the coordinate axis. Details can be found again in the monograph [53]. For example, the five- and four- dimensional volume elements are calculated to be

$$V_5 = \frac{1}{12}\pi^2 c^6 (sh\frac{4r}{c} - 8sh\frac{2r}{c} + 12\frac{r}{c}) \quad , \tag{10.17}$$

$$V_4 = \pi c^3 (sh\frac{2r}{c} - \frac{2r}{c}) \qquad , \tag{10.18}$$

where r denotes the natural (euclidean) length and $c = \frac{1}{k}$ is the unit length parameter for the Lobachevsky space, which enters not only the expressions (10.17 - 10.18), but also the exponential factor $e^{-2kr-\Phi}$. This factor should be present also in the $(d+4)$- dimensional analogue of the metric (10.12). Unfortunately, in most of the existing papers on theories with extra dimensions, the origin and meaning of the parameter k is not clarified, due to which we point out this here.

Since in the limit $c \to \infty$ the usual Euclidean geometry is recovered [54], then the above formulaes would give the volumes of the five- and of the four- dimensional (Euclidean) spheres respectively

$$V_5 = \frac{8}{15}\pi^2 r^5 \quad ; \quad V_4 = \frac{4}{3}\pi r^3 (1 + \frac{1}{5}\frac{r^2}{c^2} + ...) \ = \ \frac{4}{3}\pi r^3 \quad . \tag{10.19}$$

The volumes of the n- dimensional (Euclidean) spheres for $n = 2\lambda$ and $n = 2\lambda + 1$ [53]

$$V_{2\lambda} = \frac{\pi^{\lambda}}{\lambda!} r^{2\lambda} \quad ; \quad V_{2\lambda+1} = \frac{2^{\lambda+1}\pi^{\lambda}}{(2\lambda+1)(2\lambda-1)....3.1} r^{2\lambda+1} \tag{10.20}$$

can also be derived in the limit $c \rightarrow \infty$ from the (recurrent) formulae for the n- dimensional hyperbolic volume

$$V_n = \frac{2\pi c^2}{(n-1)} \left[\frac{P_{n-4}}{(n-2)} c^{n-2} sh^{n-2}\frac{r}{c} ch\frac{r}{c} - V_{n-2} \right] \quad , \tag{10.21}$$

where

$$P_{2\lambda} = \frac{2\pi^{\lambda+1}}{\lambda!} \quad ; \quad P_{2\lambda+1} = \frac{2^{\lambda+2}\pi^{\lambda+1}}{(2\lambda+1)(2\lambda-1)(2\lambda-3)....3.1} \quad . \tag{10.22}$$

Therefore, only in the flat (Euclidean) $(4 + d)$- dimensional space, which is a product of a 4- dimensional Minkowski space ($\mu, \nu = 1, 2, .., 4$) and a flat d- (extra)dimensional space [52]

$$ds^2 = \eta_{\mu\nu} dX^{\mu} dX^{\nu} - r^2 d\Omega^2_{(d)} \quad , \tag{10.23}$$

one can factorize the volume element in the $(4 + d)$ Einstein - Hilbert action (assuming also that $R^{(4+d)} = R^{(4)}$) as

$$S_{4+d} = -M_*^{d+2} \int d^{4+d}X \sqrt{g^{(4+d)}} R^{(4+d)} = -M_*^{(d+2)} \int d\Omega_{(d)} r^d \int d^4X \sqrt{g^{(4)}} R^{(4)} \quad . \tag{10.24}$$

It is clear however from expressions (10.17-18), (10.21) that in the general case of a multidimensional non-euclidean (Lobachevsky) space such a factorization of the volume element is impossible. Even in the limit of small ratios $\frac{r}{c}$, possible corrections to the volume element (see f.(10. 19)) have to be taken into account and therefore, the non-euclidean geometry would "induce" correction terms in the relation between the gravitational couplings.

But again in the flat euclidean case, from (10.24) in [52] the relation (10.15) $M^2_{pl} = M_*^{d+2} V_{(d)}$ between the gravitational couplings has been correctly found. Remarkably, the same matching relation has been found in [47] on the base of a different approach, namely the $(4 + d)$- dimensional Gauss law

$$\oint_{surf.C} F \; dS = S_{(3+d)} G_{N(4+d)} \times Mass \quad , \tag{10.25}$$

where it is important to point out that $S_{(3+d)} \equiv \frac{2\pi^{\frac{3+d}{2}}}{\Gamma(\frac{3+d}{2})}$ is the surface area of the euclidean (!) unit sphere in $(3 + d)$- dimensions. So again, provided that the Newton's force law in the $(4 + d)$- dimensional space is

$$F_{(4+d)}(r) = G_{N(4+d)} \frac{m_1 m_2}{r^{n+2}} \quad , \tag{10.26}$$

the s. c. "matching relation" has been recovered. This really proves that in a $(4+d)$-dimensional flat space the correct Newton's law is indeed given by expression (10.26). It should be stressed that there is a perfect matching between all results due to the initial assumption about the flat (Euclidean) metric.

10.2.3. COORDINATE TRANSFORMATIONS WITH THE LOBACHEVSKY CONSTANT IN "WARP" TYPE OF METRICS

Now let us turn to the other frequently used case [55, 56] of a $5D$- metric with an exponentially suppressed "warp" factor in front of the flat $4D$ Minkowski metric $(0 \leq y \leq \pi r_c)$

$$ds^2 = e^{-2\widetilde{k}|y|}\eta_{\mu\nu}dX^{\mu}dX^{\nu} + dy^2 \tag{10.27}$$

and for the moment it shall be assumed that $\widetilde{k}$ is different from the constant $k = \frac{1}{c}$. The five - dimensional effective action can be factorized as [55]

$$S_{eff} = \int d^4X \int\limits_0^{\pi r_-} dy 2M^3 r_c e^{-2\widetilde{k}|y|}\sqrt{g}R \tag{10.28}$$

and the metric (10.27) is chosen so that the 5- dimensional Ricci scalar curvature is equal to the 4- dimensional Minkowski one. From (10.28), the matching relation between the gravitational couplings is obtained to be [55]

$$M_{pl}^2 = 2M^3 \int\limits_0^{\pi r_-} dy e^{-2\widetilde{k}|y|} = \frac{M^3}{\widetilde{k}}(1 - e^{-2\widetilde{k}r_-\pi}) \quad . \tag{10.29}$$

If one gives up the assumption that the four- dimensional scalar curvature is equal to the five- dimensional one, but just fixes a given scalar curvature and keeps the exponential prefactor in the $4D$ Minkowski part of the metric, the approach of the cubic algebraic equation in this paper might be used to find the metric component in front of dy^2, which will be nothing else, but the contravariant metric tensor component $\widetilde{g}^{55} = dX^5 dX^5$. For this special case, it is not obligatory to find the solution in terms of elliptic functions or to express it in complex coordinates - as already mentioned, parametrization with the Weierstrass function is just one of the ways for solving a cubic algebraic equation.

Let us note that the result of the integration in (10.28 - 29) will not be coordinate independent. In other words, if we map the $4D$ Minkowski part of the metric with an exponentially "damped" prefactor into a flat Minkowski $4D$ metric without the exponential prefactor, this would result in the appearence of an exponentially growing prefactor in front of the dy^2 part of the metric (10.27). To illustrate this, let us use a coordinate transformation, similar to the one, used in [57]

$$X^1 = a \; ch\frac{\rho}{c} \quad ; \quad X^2 = b \; sh\frac{\rho}{c}\sin\Theta\cos\varphi \quad , \tag{10.30}$$

$$X^3 = b\ sh\frac{\rho}{c}\sin\Theta\sin\varphi \quad ; \quad X^4 = b\ sh\frac{\rho}{c}\cos\Theta \quad . \tag{10.31}$$

The signature of the Minkowski space is $(+,-,-,-)$, i.e. $\eta_{\mu\nu} = (+1,-1,-1,-1)$, c and r are the Lobachevsky constant (natural distance unit in the Lobachevsky space) and the euclidean distance respectively and ρ is the distance in the Lobachevsky space, related to the euclidean distance r by the formulae

$$r = c\ sh\frac{\rho}{c} \quad . \tag{10.32}$$

The constants a and b are to be chosen so that a flat Minkowski metric without any prefactor is obtained. In fact, if $\widetilde{k} = \frac{1}{c}$, the metric (10.27) will be exactly the one, known from Lobachevsky geometry. Now we shall establish the physical meaning of the relation $\widetilde{k} = \frac{1}{c}$, but in reference to theories with extra dimensions. An elementary introduction into Lobachevsky geometry can be found in [58] and a more comprehensive and detailed exposition - in [59].

For the choice $a = b = ce^{\widetilde{k}|y|}$ and after applying the transformations (10.30 - 31), the metric (10.27) can be rewritten as

$$ds^2 = -d\rho^2 - c^2 sh^2\frac{\rho}{c}d\Theta^2 - c^2 sh^2\frac{\rho}{c}\sin^2\Theta d\varphi^2 + (c\widetilde{k})^2 e^{2\widetilde{k}|y|}dy^2 \quad . \tag{10.33}$$

Obviously, the first three terms give the unit lenght element in the Lobachevsky space, which in the limit $c \rightarrow \infty$ (then from (10.32 $\rho \rightarrow r$) gives the usual euclidean lenght element $dr^2+r^2(d\Theta^2+\sin^2\Theta d\varphi^2)$ in spherical coordinates (r,Θ,φ). Most interesting is the last term in (10.33) - it goes to infinity if $\mid y \mid \rightarrow \infty$ and in the limit $c \rightarrow \infty$ (when $c \neq \frac{1}{k}$), but it tends to 1 in the limit $c \rightarrow \infty$ and when $\widetilde{k} = \frac{1}{c}$, which is physically reasonable because a flat euclidean geometry is obtained, as it should be. Thus the exponential increase of the "extra - dimensional" distance, when $c \neq \frac{1}{k}$, can be regarded as an effect of the non-euclidean nature of space-time. Indeed, it is physically unacceptable to take the limit $\widetilde{k} = \frac{1}{c} \rightarrow 0$, because if the five - dimensional Planck mass is assumed to be not very far from the electroweak scale $M_W \approx TeV$, then a fine- tuning $\widetilde{k}\ r_c \approx 50$ is needed [49].

Of course, the same approach may be applied to more complicated models with an arbitrary "warp" exponential factor and $(D-4)$ compact non-flat extra- dimensional spacetime [60, 61]

$$ds^2 = g_{ab}(\mathbf{X})dX^a dX^b = 2e^{2A(y)}\eta_{\mu\nu}dX^\mu dX^\nu + h_{ij}(y)dy^i dy^j \quad , \tag{10.34}$$

where $(a,b) = 1,2,...D$; $(\mu,\nu) = 1,..,4$; $(i,j) = 5,...,D$. The transformations (10.30 - 31) can again be used (with $a = b = \widetilde{k}\ e^{-A(y)}$) and an expression for the warp factor $A(y)$ can be found so that the metric tensor components h_{ij} of the extra- dimensional space are left unchanged. In principle, the motivation for different warp factors comes from M- theory (see [62] for a recent review), where for example three warp factors might be allowed: one for the $2+1$ dimensional spacetime and two for the internal manifold.

10.2.4. ALGEBRAIC EQUATIONS IN $4D$ SCHWARZSCHILD BLACK HOLES IN HIGHER DIMENSIONAL BRANE WORLDS

Now suppose that the metric (10.27) does contain a flat Minkowski $4D$ space, but a $4D$ black hole instead

$$ds^2 = e^{-2\widetilde{k}|y|} g_{\mu\nu} dX^{\mu} dX^{\nu} + dy^2 \quad , \tag{10.35}$$

where

$$g_{\mu\nu} dX^{\mu} dX^{\nu} = -(1 - \frac{2M}{r}) dt^2 + (1 - \frac{2M}{r})^{-1} dr^2 + \\ + r^2 (d\Theta^2 + \sin^2 \Theta d\varphi^2) \quad . \tag{10.36}$$

For such a model [63, 64], if a negative tension brane is introduced at a distance $y = l < \infty$, the five dimensional BH singularity will have a finite size and a black tube will extend into the bulk, thus interpolating between the two black holes. In analogy with the previous considerations, an interesting problem is to find whether there exists a transformation of the type

$$X^1 \equiv t = e^{\widetilde{k}|y|} g_1(\rho, \widetilde{\Theta}, \widetilde{\varphi}) \ ch\frac{\rho}{c} \quad ; \quad X^2 \equiv r = e^{\widetilde{k}|y|} g_2(\rho, \widetilde{\Theta}, \widetilde{\varphi}) \ sh\frac{\rho}{c} \sin \widetilde{\Theta} \cos \widetilde{\varphi} \quad , \tag{10.37}$$

$$X^3 \equiv \Theta = e^{\widetilde{k}|y|} g_3(\rho, \widetilde{\Theta}, \widetilde{\varphi}) \ sh\frac{\rho}{c} \sin \widetilde{\Theta} \sin \widetilde{\varphi} \quad ; \quad X^4 \equiv \varphi = e^{\widetilde{k}|y|} \ g_4(\rho, \widetilde{\Theta}, \widetilde{\varphi}) sh\frac{\rho}{c} \cos \widetilde{\Theta} \quad , \tag{10.38}$$

so that in terms of the new coordinates again the known hyperbolic spherical element (without any exponential prefactor) will be obtained plus a term in front of the extra-dimensional coordinate y. **If the functions g_i include also the variable $\widetilde{t}$, i.e. $g_i = g_1(\rho, \widetilde{t}, \widetilde{\Theta}, \widetilde{\varphi})$, it will be interesting also to see does a transformation exist, so that the black hole in terms of the coordinates $(\rho, \widetilde{t}, \widetilde{\Theta}, \widetilde{\varphi})$ in the limit $c \rightarrow \infty$ will give again the known Schwarzschild black hole element (10.36)?**

A possible application of the formalism in this paper is related to the Riemann scalar curvature invariant $R_{ABCD}R^{ABCD}$ [63], which for the background metric (10.36) and using the conventional contravariant metric components g^{ij} is calculated to be [63]

$$R_{ABCD} R^{ABCD} = 40k^4 + \frac{48M^2 e^{4k|y|}}{r^6} \quad . \tag{10.39}$$

This expression contains an important physical information - it diverges at the black hole singularity at $r = 0$ and also at the ADS horizon at $\mid y \mid \rightarrow \infty$. The elimination of this singulariry (i. e. giving it a finite size) is the main motivation for introducing the second, negative tension brane at a distance $y = L$. But even in the case of a single brane configuration the presence of a singularity is essential since there might be a non-vanishing energy flow into the bulk singularities, which is not desirable. Such an energy flow will exist if the limit [63]

$$\lim_{|y| \rightarrow \infty} \sqrt{-g} J^y_{(\mu)} = \lim_{|y| \rightarrow \infty} \sqrt{-g} T^{yN} K^{(\mu)}_N \tag{10.40}$$

is non- zero, where $J^{y}_{(\mu)}$ and T^{yN} are the current and the energy- momentum tensor of a massless scalar field and $K^{(\mu)}_{N} = e^{2A}\delta^{\mu}_{M}$ $(\mu = t, \Theta, \varphi)$ is the Killing vector for the BH metric (10.35 - 36).

Therefore, it is essential to check whether the presence of the singularity in (10.39) and of the zero energy flow in (10.40) will be confirmed if the same scalar curvature R will be obtained by contracting the Riemann tensor with another contravariant metric tensor field $\widetilde{g}^{ij}$ such that

$$R = g^{AC}g^{BD}R_{ABCD} = \widetilde{g}^{AC}\widetilde{g}^{BD}\widetilde{R}_{ABCD} \qquad (10.41)$$

where $\widetilde{R}_{ABCD}$, following the notation and approach in Sect. II, is the **modified Riemann tensor**

$$\widetilde{R}_{ABCD} \equiv \frac{1}{2}(g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC}) + g_{np}(\widetilde{\Gamma}^{n}_{BC}\widetilde{\Gamma}^{p}_{AD} - \widetilde{\Gamma}^{n}_{BD}\widetilde{\Gamma}^{p}_{AC}) =$$

$$= \frac{1}{2}(....) + g_{np}g_{rs}g_{qt}\widetilde{g}^{ns}\widetilde{g}^{pt}(\Gamma^{n}_{BC}\Gamma^{p}_{AD} - \Gamma^{n}_{BD}\Gamma^{p}_{AC}) \quad . \qquad (10.42)$$

If the scalar curvature R, the connection Γ^{C}_{AB} are calculated from the initially given metric, equation (10.41) can be treated as an **fourth- order** algebraic equation in respect to the components $\widetilde{g}^{AB}$ and as an **eight- order** algebraic equation in respect to the variables dX^{A}, if again the factorization $\widetilde{g}^{AB} = dX^{A}dX^{B}$ is used. This example clearly shows the necessity to go beyond the assumption about the contravariant metric factorization. But on the other hand, even if the factorization assumption is used, the same scalar curvature can be obtained by contracting the (modified) Ricci tensor with the contravariant metric tensor $\widetilde{g}^{AB}$, i. e. $R = \widetilde{g}^{AB}R_{AB}$, which was in fact the cubic algebraic equation, investigated in the previous sections.

But there is also one more way for obtaining the scalar curvature R - by assuming that the following algebraic equation with the usual Riemann tensor components holds

$$R = \widetilde{g}^{AC}\widetilde{g}^{BD}R_{ABCD} \quad . \qquad (10.43)$$

Fortunately, this equation is second order in respect to $\widetilde{g}^{AB}$ and fourth order in respect to dX^{A} and moreover, it **does no**t contain any derivatives of the components $\widetilde{g}^{AB}$ and dX^{A}, unlike the investigated cubic algebraic equation, which contains such derivatives and consequently a special approach is needed for solving such an equation.

Let us now assume that in the framework of the factorization assumption, **both** equations (10.41) and (10.43) are fulfilled. Then the fulfillment of these equations is a necessary condition for the preservation of the scalar curvature invariant because

$$R_{ABCD}R^{ABCD} = R_{ABCD}\widetilde{g}^{Ai}\widetilde{g}^{Bj}\widetilde{g}^{Ck}\widetilde{g}^{Dl}\widetilde{R}_{ijkl} = \qquad (10.44)$$

$$= \left(R_{ABCD}dX^{A}dX^{B}dX^{C}dX^{D}\right)\left(\widetilde{R}_{ijkl}dX^{i}dX^{j}dX^{k}dX^{l}\right) = \qquad (10.45)$$

$$= \left(R_{ABCD}\widetilde{g}^{AC}\widetilde{g}^{BD}\right)\left(\widetilde{R}_{ijkl}\widetilde{g}^{ik}\widetilde{g}^{jl}\right) = R^{2} \quad . \qquad (10.46)$$

Motivated by the necessity to investigate lower degree algebraic equations, one may take equation (10.43) and also the equation

$$\widetilde{g}^{AC}\widetilde{g}^{BD}\widetilde{R}_{ABCD} - \widetilde{g}^{AC}\widetilde{g}^{BD}R_{ABCD} = 0 \quad . \tag{10.47}$$

A subclass of solutions of this equation will be represented by the algebraic equation

$$\widetilde{g}^{BD}\widetilde{R}_{ABCD} - \widetilde{g}^{BD}R_{ABCD} = 0 \tag{10.48}$$

(cubic in respect to $\widetilde{g}^{AB}$ and of sixth order in respect to dX^{A}) and another, more restricted class of solutions - by the equation

$$\widetilde{R}_{ABCD} - R_{ABCD} = 0 \quad , \tag{10.49}$$

which is quadratic in $\widetilde{g}^{AB}$ and quartic in respect to dX^{A}. **Therefore, even in such a complicated case, the investigation of the intersection varieties of the two quartic equations (10.43) and (10.49), written respectively as (again, it shall be used that $\widetilde{\Gamma}_{ij}^{k} = dX^{k}dX^{s}g_{rs}\Gamma_{ij}^{r}$)**

$$dX^{A}dX^{B}dX^{C}dX^{D}R_{ABCD} - R = 0 \tag{10.50}$$

and

$$\begin{aligned} &g_{np}g_{rs}g_{qt}(\Gamma^{r}_{BC}\Gamma^{q}_{AD} - \Gamma^{r}_{BD}\Gamma^{q}_{AC})dX^{n}dX^{s}dX^{p}dX^{t} - \\ &-g_{np}(\Gamma^{n}_{BC}\Gamma^{p}_{AD} - \Gamma^{n}_{BD}\Gamma^{p}_{AC}) = 0 \quad , \end{aligned} \tag{10.51}$$

may give some solutions for the contravariant metric tensor components $\widetilde{g}^{AB} = dX^{A}dX^{B}$, which will preserve both the scalar curvature and the scalar curvature invariant. Respectively, if only the scalar curvature R is to be preserved, one may find the solutions of the algebraic equation (10.43) and then substitute them in the expression for the scalar curvature invariant $R_{ABCD}R^{ABCD}$.

It is clear also that if one takes only equation (10.41) $R = \widetilde{g}^{AC}\widetilde{g}^{BD}\widetilde{R}_{ABCD}$ and not equation (10.43), from (10.44- 46) one may obtain not the equality $R_{ABCD}R^{ABCD} = R^{2}$, but an fourth- order algebraic equation in respect to dX^{A} for the preservation of the scalar curvature invariant

$$R.R_{ABCD}dX^{A}dX^{B}dX^{C}dX^{D} - R_{ABCD}R^{ABCD} = 0 \quad . \tag{10.52}$$

But this is not the only possibility. One may take also **only** equation (10.43) $R = \widetilde{g}^{AC}\widetilde{g}^{BD}R_{ABCD}$ and disregard equation (10.41). Then the resulting algebraic equation from (10.44 - 46) will be

$$\begin{aligned} &\frac{1}{2}(g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC})dX^{A}dX^{B}dX^{C}dX^{D} + \\ &+g_{np}g_{rs}g_{qt}(\Gamma^{r}_{BC}\Gamma^{q}_{AD} - \Gamma^{r}_{BD}\Gamma^{q}_{AC})dX^{A}dX^{B}dX^{C}dX^{D}dX^{n}dX^{s}dX^{p}dX^{t} - \\ &-R_{ABCD}R^{ABCD} = 0 \quad . \end{aligned} \tag{10.53}$$

This equation is of eight order and due to the presence of the last scalar curvature invariant term it is impossible to find subclasses of solutions of (lower - order) algebraic equations, as in the case of eq. (10.47).

10.2.5. COMPACTIFICATION RADIUS AND SCALAR FIELD EQUATION IN $4D$ SCHWARZSCHILD BLACK HOLES IN HIGHER DIMENSIONAL BRANE WORLDS

In theories with extra dimensions, for example $(4+n)$- dimensional Schwarzschild black hole [64, 65, 66, 67]

$$ds^2 = -h(r)dt^2 + h^{-1}(r)dr^2 + r^2 d\Omega^2_{n+2} \tag{10.54}$$

with

$$h(r) = 1 - \left(\frac{r_H}{r}\right)^{n+1} \tag{10.55}$$

(r_H- the horizon radius) it is important to distinguish between distances $r \ll R_1$ (R_1- the compactification radius), when the BH is a $(4+n)$- dimensional one, and distances $r \gg R_1$, when the BH metric goes over to the usual four dimensional Schwarzschild metric

$$ds^2 = -(1 - \frac{2M}{M^2_{Hr}})dt^2 + (1 - \frac{2M}{M^2_{Hr}})^{-1}dr^2 + r^2 d\Omega^2 \quad . \tag{10.56}$$

However, when solving the scalar wave equation $g^{IJ}\Phi_{I;J} = 0$, there is no way to introduce the scale factor R_1 in the solution of the scalar equation - in such a case the scalar field behaviour can be compared in the transition from one limit to another.

The use of the more general contravariant tensor $\widetilde{g}^{ij}$ gives the opportunity to introduce such a scale factor. Let us first note that

$$g_{AB}\widetilde{g}^{BC} = l^C_A(\mathbf{x}) \quad \Rightarrow \quad \widetilde{g}^{BC} = l^B_D g^{DC} \quad , \tag{10.57}$$

where A, B, C, D concretely in this case will denote the $(4+n)$- dimensional indices, μ, ν- only the four- dimensional indices and i, j, k denote the indices of the additional n-dimensional space. Then the metric can be represented as

$$ds^2 = g_{AB}dX^A dX^B = g_{\mu\nu}dX^\mu dX^\nu + \sum_{i=5}^{n+4} l^i_i = ds^2_{(4)} + nR_1 \tag{10.58}$$

where it has been assumed that $l^i_i = R_1$ for all i. Consequently, some of the components $\widetilde{g}^{jB}$ of the contravariant metric tensor can be expressed as

$$\widetilde{g}^{jB} = l^j_\nu g^{\nu B} + l^j_i g^{iB} = l^j_\nu g^{\nu B} + R_1 g^{jB} + l^j_{\substack{i \\ i \neq j}} g^{iB} \tag{10.59}$$

and evidently the solutions of the scalar wave equation will depend on the compactification radius R_1.

10.2. 6. A COMPLIMENTARY PROPOSAL FOR HIGGS MASS GENERATION IN THEORIES WITH TWO THREE - BRANES

Closely related to the above problem about the contravariant metric tensor components as coupling constants is the problem about **Higgs mass generation** in theories with two branes [46, 56] - the so called **"hidden"** and **"visible**" branes at the orbifold fixed points $\Phi = 0$ and $\Phi = \pi$ (the metric is again (10.10)). These three branes couple to the four dimensional components $G_{\mu\nu}$ of the bulk metric as [46]

$$g_{\mu\nu}^{vis}(X^{\mu}) = G_{\mu\nu}(X^{\mu}, \Phi = \pi) \quad ; \quad g_{\mu\nu}^{hid}(X^{\mu}) = G_{\mu\nu}(X^{\mu}, \Phi = 0) \tag{10.60}$$

and the action includes the gravity part plus the action the action for the visible and hidden branes and also the part of the action for the fundamental Higgs field

$$S_{vis} = \int d^4X\sqrt{-g}_{vis}\left[g_{vis}^{\mu\nu}D_{\mu}H^{+}D_{\nu}H - \lambda\left(\mid H \mid^2 - v_0^2\right)^2\right] \quad , \tag{10.61}$$

where v_0 is the vacuum expectation value (VEV) for the Higgs field H, λ is a coupling constant [46]. Similar coupling of the contravariant metric tensor components to a gauge field can be found also in radion cosmology theories [68]. Since $g_{\mu\nu}^{vis} = e^{-2kr_{-}\pi}g_{\mu\nu}$, it is believed that by a proper normalization of the fields one can determine the physical masses. In particular, if the Higgs field wave function is normalized as $H \rightarrow e^{kr_{-}\pi}H$, then

$$S_{vis} = \int d^4X\sqrt{-g}\left[g^{\mu\nu}D_{\mu}H^{+}D_{\nu}H - \lambda\left(\mid H \mid^2 - e^{-2kr_{-}\pi}v_0^2\right)^2\right] \quad . \tag{10.62}$$

Therefore, since $v = e^{-2kr_{-}\pi}v_0$, any mass m_0 on the visible three- brane in the fundamental higher- dimensional theory will correspond to a physical mass

$$m = e^{-kr_{-}\pi}m_0 \quad , \tag{10.63}$$

"measured" with the metric $g^{\mu\nu}$ in the effective Einstein- Hilbert action. If $kr_c \approx 50$ (i.e. $e^{kr_{-}\pi} \approx 10^{15}$), this is the physical mechanism that is supposed to produce TeV physical mass scales from mass parameters around the Planck scale $\approx 10^{19}$ GeV.

In the context of the developed approach in this paper, now it shall be shown that the above physical mechanism of generation of TeV mass scales may turn out to be more complicated and diverse. Namely, for a given scalar curvature, there will be a multitude of contravariant metric tensors, thus suggesting that the possibilities for the mass scales will be much more in number.

Following the two- dimensional approach, the contravariant metric tensor components $\widetilde{g}^{\mu\nu}$ can be written as

$$\widetilde{g}^{\mu\nu} = dX^{\mu}dX^{\nu} = F_{\mu}(\mathbf{X}(z,v), \Phi(z,v), z)F_{\nu}(\mathbf{X}(z,v), \Phi(z,v), z) \quad , \tag{10.64}$$

from where the (contravariant) metric on the visible brane can be expressed as

$$\widetilde{g}^{\mu\nu}_{vis} = L_2(z,v)\widetilde{g}^{\mu\nu} \quad , \tag{10.65}$$

where

$$L_2(z,v) \equiv \frac{F_{\mu}(\mathbf{X}(z,v),\Phi(z,v)=\pi,z)F_{\nu}(\mathbf{X}(z,v),\Phi(z,v)=\pi,z)}{F_{\mu}(\mathbf{X}(z,v),\Phi(z,v),z)F_{\nu}(\mathbf{X}(z,v),\Phi(z,v),z)} \quad . \tag{10.66}$$

Formulaes (10.65 - 66) have been derived as a ratio of the "visible" and the usual contravariant metric components for each fixed indices $(\mu,\nu)=(\mu_1,\nu_1)$ and for the moment, **without assuming** that the points on the complex plane, for which $\Phi(z_0,v_0)=\pi$, are known. Further it shall be shown how the calculation will be modified if these points are assumed to be known.

The transition from the four- dimensional variables $d^4X = dX_1dX_2dX_3dX_4$ to the two- dimensional complex variables (z,v) can be performed by using the formulae

$$d^4X = \sum_{1\leq i_1<i_k\leq 4} \det \begin{Vmatrix} \frac{\partial X_{i_1}}{\partial z} & \frac{\partial X_{i_k}}{\partial v} \\ \frac{\partial X_{i_k}}{\partial z} & \frac{\partial X_{i_k}}{\partial v} \end{Vmatrix} dz\wedge dv = L_3(z,v)dz\wedge dv \quad , \tag{10.67}$$

but since we are interested in rescaling only the Higgs field and the contravariant metric as

$$H \rightarrow \widetilde{H}f \quad ; \quad v_0 \rightarrow \widetilde{v}_0 \qquad (f - a\ function) \quad , \tag{10.68}$$

the change of variables in the volume integration is not necessary to be taken into account. Next it is necessary to find how the volume element $\sqrt{-g_{vis}}$ of the visible brane can be expressed through the volume element $\sqrt{-g}$ in terms of the metric (10.10). It can easily be calculated that

$$\sqrt{-g} = \sqrt{K_1(\Phi,\frac{\partial\Phi}{\partial z},\frac{\partial\Phi}{\partial v},\frac{\partial X^{\mu}}{\partial z},\frac{\partial X^{\mu}}{\partial z}) + e^{-4kr_-\Phi}K_2(\frac{\partial X^{\mu}}{\partial z},\frac{\partial X^{\mu}}{\partial v})} \quad , \tag{10.69}$$

where

$$K_1 \equiv r_c^2e^{-2kr_-\Phi}[(\frac{\partial\Phi}{\partial z})^2(\frac{\partial X^1}{\partial v})^2 + (\frac{\partial\Phi}{\partial v})^2(\frac{\partial X^1}{\partial z})^2 - (\frac{\partial\Phi}{\partial z})^2\sum_{i=2}^{4}(\frac{\partial X^i}{\partial v})^2 -$$

$$-(\frac{\partial\Phi}{\partial v})^2\sum_{i=2}^{4}(\frac{\partial X^i}{\partial z})^2] + 3r_c^4(\frac{\partial\Phi}{\partial z})^2(\frac{\partial\Phi}{\partial v})^2 - 8r_c^2\frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial v}\frac{\partial X^1}{\partial z}\frac{\partial X^1}{\partial v}e^{-2kr_-\Phi} +$$

$$+8r_c^2e^{-2kr_-\Phi}\frac{\partial\Phi}{\partial z}\frac{\partial\Phi}{\partial v}\sum_{i=2}^{4}\frac{\partial X^i}{\partial z}\frac{\partial X^i}{\partial v} \quad , \tag{10.70}$$

$$K_2 \equiv 8\frac{\partial X^1}{\partial z}\frac{\partial X^1}{\partial v}\sum_{i=2}^{4}\frac{\partial X^i}{\partial z}\frac{\partial X^i}{\partial v} - \left(\frac{\partial X^1}{\partial z}\right)^2\sum_{i=2}^{4}\left(\frac{\partial X^i}{\partial v}\right)^2 - \left(\frac{\partial X^1}{\partial v}\right)^2\sum_{i=2}^{4}\left(\frac{\partial X^i}{\partial z}\right)^2 -$$

$$-3\left(\frac{\partial X^1}{\partial z}\right)^2\left(\frac{\partial X^1}{\partial v}\right)^2+\left(\sum_{i=2}^{4}\frac{\partial X^i}{\partial z}\right)^2\left(\sum_{i=2}^{4}\frac{\partial X^i}{\partial z}\right)^2-$$

$$-4\left(\sum_{i=2}^{4}\frac{\partial X^i}{\partial z}\frac{\partial X^i}{\partial v}\right)^2 \quad . \tag{10.71}$$

Setting up $\Phi(z,v)=\pi$ (note that then $K_1=\pi$), one obtains

$$\sqrt{-g}_{vis}=L_1(\Phi,X^\mu,\sqrt{-g})\sqrt{-g} \quad , \tag{10.72}$$

where

$$L_1(\Phi,X^\mu,\sqrt{-g})\equiv\frac{e^{-2kr_-\pi}}{e^{-2kr_-\Phi}}\sqrt{1-\frac{K_1(\Phi,X^\mu)}{(\sqrt{-g})^2}} \quad . \tag{10.73}$$

Unlike the previously discussed in [46] case, when the "visible" volume element is represented as a product of some factor (constant), multiplying the volume element $\sqrt{-g}$ (i.e. $\sqrt{-g}_{vis}=e^{4kr_-\pi}\sqrt{-g}$), the present case might seem to be quite different, since the function L_1 depends again on $\sqrt{-g}$. However, it shall be proved below that by requiring the action of the "visible" brane to remain unchanged after the rescaling (10.68), still such a possibility will exist, but in a more general form. Indeed, after the rescaling (10.68) $H\rightarrow\widetilde{H}f \quad ; \quad v_0\rightarrow\widetilde{v}_0$ the action (10.61) becomes (written in the two - dimensional coordinates (z,v))

$$S_{vis}=\int dzdv\sqrt{-g}L_3L_1\left[g^{\mu\nu}L_2\ f^2D_\mu\widetilde{H}^+D_\nu\widetilde{H}-\lambda f^4\left(\mid\widetilde{H}\mid^2-\widetilde{v}_0^2\right)^2\right]+$$

$$+\int dzdv\sqrt{-g}\ L_3L_{add} \quad , \tag{10.74}$$

where

$$L_{add}\equiv L_2g^{\mu\nu}[\mid\widetilde{H}\mid^2A_\nu\partial_\mu\mid f\mid^2+\mid\widetilde{H}\mid^2A_\nu\partial_\mu f^+\partial_\nu f+$$

$$+\widetilde{H}^+f\ \partial_\mu f^+\partial_\nu\widetilde{H}+\widetilde{H}\ f^{\ +}\partial_\mu f\ \partial_\nu\widetilde{H}^+ \tag{10.75}$$

and the covariant derivative D_μ is expressed as $D_\mu=\partial_\mu+A_\mu$. Clearly, the visible brane actions before and after the rescaling will remain unchanged if

$$L_1L_2f^2=1 \quad ; \qquad L_1f^4=1 \tag{10.76}$$

and

$$L_{add}=0 \quad . \tag{10.77}$$

The first two relations (10.76) give

$$f=\pm(L_2)^{\frac{1}{2}}=\pm(L_1)^{-\frac{1}{6}} \quad , \tag{10.78}$$

which can be rewritten as

$$\frac{1}{L_2^3} = \frac{e^{-2kr_{-}\pi}}{e^{-2kr_{-}\Phi}} \sqrt{1 - \frac{K_1(\Phi, X^\mu)}{(\sqrt{-g})^2}} \qquad , \tag{10.79}$$

from where the function $K_1(\Phi, X^\mu)$ can be expressed and substituted into expression (10.69) for $\sqrt{-g}$. Thus one obtains

$$\sqrt{-g} = L_2^3 e^{-2kr_{-}\pi} \sqrt{K_2(X^\mu)} \qquad . \tag{10.80}$$

From (10.69) for $\Phi(z, v) = \pi$ one can easily derive

$$\sqrt{-g}_{vis} = \sqrt{e^{-4kr_{-}\pi}}.\sqrt{K_2(X^\mu)} = \frac{1}{L_2^3}\sqrt{-g} \qquad . \tag{10.81}$$

Therefore, even in the more general case of contravariant metric tensor, different from the inverse one, there is a relation similar to $\sqrt{-g}_{vis} = e^{-4kr_{-}\pi}\sqrt{-g}$, but with the function $\frac{1}{L_2^3}$, multiplying the volume element. Let us remind that for performing the calculation it was sufficient to know the function $\Phi(z, v)$ as a solution of the system of nonlinear differential equations, but not the points $(z_0^{(l)}, v_0^{(l)})$, at which $\Phi(z = z_0^{(l)}, v = v_0^{(l)}) = \pi$. Consequently, in the final result (10.81) one **cannot set up**

$$\sqrt{-g}_{vis}\left(\mathbf{X}(z = z_0^{(l)}, v = v_0^{(l)}), \Phi = \pi\right) = \frac{1}{L_2^3(z = z_0^{(l)}, v = v_0^{(l)}, \Phi = \pi)}\sqrt{-g} \qquad . \tag{10.82}$$

Then **to any mass m_0 on the visible three- brane would correspond a single physical mass, "measured" with the metric $g^{\mu\nu}$**

$$m^{(l)} = m_0 f = m_0 \sqrt[4]{L_2^{(l)}(z = z_0^{(l)}, v = v_0^{(l)}, \Phi = \pi)} \qquad , \tag{10.83}$$

i. e. there is no degeneracy of masses. The corresponding additional condition (10.77) $L_{add} = 0$ can be written as

$$L_{add} = f^2 \ \partial_\mu \ln f \ [2 \mid \widetilde{H} \mid^2 A_\nu + 2\widetilde{H}^+ + \widetilde{H}^2 \ \ \partial_\nu \left(\frac{\widetilde{H}^+}{\widetilde{H}}\right) +$$

$$+2 \mid \widetilde{H} \mid^2 \partial_\nu(\ln f) \ - \widetilde{H}^2 \partial_\nu(\ln f)] = 0 \qquad , \tag{10.84}$$

from where the trivial case $f = const$ is obtained from $\partial_\mu \ln f = 0$.

Let us now see how the above approach will change if the points $(z_0^{(l)}, v_0^{(l)})$ on the complex plane, at which the equation $\Phi(z = z_0^{(l)}, v = v_0^{(l)}) = \pi$ holds, are considered to be known. The function $L_2(z, v)$ in the ratio of $g_{vis}^{\mu\nu}$ and $g^{\mu\nu}$ will be different and will be denoted as $\widetilde{L}_2(z, v)$

$$\widetilde{L}_2(z, v) \equiv \frac{F_\mu(\mathbf{X}(z_0^{(l)}, v_0^{(l)}), \Phi = \pi, z_0^{(l)}) F_\nu(\mathbf{X}(z_0^{(l)}, v_0^{(l)}), \Phi = \pi, z_0^{(l)})}{F_\mu(\mathbf{X}(z, v), \Phi(z, v), z) F_\nu(\mathbf{X}(z, v), \Phi(z, v), z)} \qquad . \tag{10.85}$$

Also from formulae (10.68) for $\Phi = \pi$ and for all points $(z, v) = (z_0^{(l)}, v_0^{(l)})$ one can obtain

$$\sqrt{-g}_{vis} = \sqrt{-g} e^{-2kr_-\pi} \sqrt{\frac{K_2^0(X^\mu(z_0^{(1)}, v_0^{(1)})}{K_1 + e^{-4kr_-\Phi} K_2(X^\mu(z,v))}} = \widetilde{L}_1(z,v) \quad , \tag{10.86}$$

which evidently is different from expression (10.59). Consequently, for this case instead of (10.79) one receives

$$\frac{1}{\widetilde{L}_2^6} = e^{-4kr_-\pi} \frac{K_2^0(X^\mu(z_0^{(1)}, v_0^{(1)})}{K_1 + e^{-4kr_-\Phi} K_2(X^\mu(z,v))} \quad , \tag{10.87}$$

from where the function K_1 can be expressed and substituted into expression (10.86) for $\sqrt{-g}_{vis}$. Taking into account again equality (10.68) for $\sqrt{-g}$, one obtains

$$\sqrt{-g}_{vis} = \sqrt{-g} \frac{1}{\widetilde{L}_2^3} = \frac{\sqrt{K_2^0}}{e^{2kr_-\pi}} \quad . \tag{10.88}$$

Therefore, the volume element of the "visible" brane is a constant, while the real volume element is $\widetilde{L}_2^3$ times the volume of the "visible" brane.

In this case, to any mass m_0 on the "visible" brane would correspond l in number physical masses, determined by the formulae

$$m^{(l)} = m_0 f^{(l)} = m_0 \sqrt{\widetilde{L}_2^{(l)}(z,v)} \quad , \tag{10.89}$$

where the function $\widetilde{L}_2(z,v)$ is given by (10.85). Therefore, there will be a degeneracy of masses.

10.2.7. TENSOR LENGTH SCALE, RESCALING AND COMPACTIFICATION IN THE LOW ENERGY ACTION OF TYPE I TEN - DIMENSIONAL STRING THEORY

Our next example of possible application of theories with covariant and contravariant metric tensors is related to the low - energy action of type I string theory in ten dimensions [47, 70, 71, 72]

$$S = \int d^{10}x \left(\frac{m_s^8}{(2\pi)^7 \lambda^2} R + \frac{1}{4} \frac{m_s^6}{(2\pi)^7 \lambda} F^2 + ... \right) = \int d^4x V_6(......) \quad , \tag{10.90}$$

where $\lambda \sim \exp(\Phi)$ is the string coupling (as remarked in [72], in the first term the coupling is λ^2, because it is generated by an world - sheet path integral on an sphere and the coupling λ in the second term - by an world - sheet path integral on the disc), m_s is the string scale, which we can identify with $m_{grav.}$. Compactifying to 4 dimensions on

a manifold of volume V_6, one can identify the resulting coefficients in front of the R and $\frac{1}{4}F^2$ terms with $M^2_{(4)}$ and $\frac{1}{g_4^2}$, from where one obtains [47]

$$M^2_{(4)} = \frac{(2\pi)^7}{V_6 m_s^4 g_4^2} \quad ; \qquad \lambda = \frac{g_4^2 V_6 m_s^6}{(2\pi)^7} \quad . \tag{10.91}$$

The physical meaning of the performed identification is that since the length scale $\sqrt{\alpha'}$ of string theory, the volume V of the (Calabi - Yau) manifold and the expectation value of the dilaton field cannot be determined experimentally, **they can be adjusted in such a way so that to give the desired values of the Newton's constant, the GUT (Grand Unified Theory) scale M_{GUT} and the GUT coupling constant [72].** It should be stressed that in the weakly coupled heterotic string theory (when there are no different string couplings $\lambda \sim \exp(2\Phi)$ and $\lambda \sim \exp(\Phi)$, but just one), the obtained bound on the Newton's constant [72] $G_N \geq \frac{\alpha^{\frac{4}{3}}_{GUT}}{M^2_{GUT}}$ is too large, but in the same paper [72] it was remarked that **"the problem might be ameliorated by considering an anisotropic Calabi - Yau with a scale $\sqrt{\alpha'}$ in d directions and $\frac{1}{M_{GUT}}$ in $(6-d)$ directions".**

Now we shall propose, in the spirit of the affine geometry approach, how such a different metric scale on the given manifold can be introduced by defining more general contravariant tensors. The key idea is that the contraction of the covariant metric tensor g_{ij} with the contravariant one $\widetilde{g}^{jk} = dX^j dX^k$ gives exactly (when $i = k$) the length interval [10]

$$l = ds^2 = g_{ij} dX^j dX^i \quad . \tag{10.92}$$

Naturally, for $i \neq k$ the contraction will give a tensor function $l_i^k = g_{ij} dX^j dX^k$, which can be interpreted as a "tensor" length scale for the different directions. In the spirit of the remark in [72], one can take for example

$$l_i^k = g_{ij} dX^j dX^k = L_1 \delta_i^k \;\; \text{for} \;\; i,j,k = 1,....,d \quad , \tag{10.93a}$$

$$l_a^b = g_{ac} dX^c dX^b = L_2 \delta_a^b \;\; \text{for} \;\; a,b,c = 1,....,6-d \quad . \tag{10.93b}$$

For simplicity and as a starting point, further we shall assume that for all indices i, j, k...

$$l_i^k = l \; \delta_i^k \quad . \tag{10.94}$$

In fact, this will be fulfilled if we assume that the contravariant metric tensor components $\widetilde{g}^{ij}$ are proportional to the usual inverse contravariant metric tensor g^{ij} with a function of proportionality $l(\mathbf{x})$, i.e. $\widetilde{g}^{ij} = l(\mathbf{x}) g^{ij}$ (it will be called a "conformal" rescaling). Also, further in the next parts of this paper it will be shown that in the more complicated case of different functions of proportionality for the different components, the tensor length scale can be calculated uniquely too. Further we shall call the function $l(\mathbf{x})$ "a length scale function".

Our next purpose will be to prove that if one imposes the requirement for invariance of the low - energy type I string action (10.90) under the "conformal" rescaling, i.e.

$$S = \int d^{10}x \left(\frac{m_s^8}{(2\pi)^7 \lambda^2} \widetilde{R} + \frac{1}{4} \frac{m_s^6}{(2\pi)^7 \lambda} \widetilde{F}^2 \right) = \int d^4x V_6 \, (....) =$$

$$= \int d^4x \left(M_{(4)}^2 R + \frac{1}{4}\frac{1}{g_4^2} F^2 \right) \quad , \qquad (10.95)$$

then **the length scale $l(x)$ will be possible to be determined from a differential equqtion in partial derivatives.** In other words, unlike the previously described in [47, 70, 71, 72] case, when the coefficients in front of R and F^2 **before** and **after** the compactification are identified, here we shall propose another approach to the same problem. Concretely, first a rescaling of the contravariant metric components shall be performed, and **after that** the compactification shall be realized, resulting again in the R.H.S. of the standard $4D$ action (10.90).

However, in principle another approach is also possible. One may start from the "unrescaled" ten - dimensional action (10.90), then perform a compactification to the four - dimensional manifold and **afterwards** a transition to the usual "unrescaled" scalar quantities R and F^2. Thus it is required that the "unrescaled" ten - dimensional effective action (10.90) (i.e. the L. H. S. of (10.90)) is equivalent to the four - dimensional effective action after compactification, but in terms of the rescaled quantities $\widetilde{R}$ and $\widetilde{F}^2$ in the R.H.S of (10.90). This can be expressed as follows

$$S = \int d^{10}x \left(\frac{m_s^8}{(2\pi)^7 \lambda^2} R + \frac{1}{4}\frac{m_s^6}{(2\pi)^7 \lambda} F^2 \right) = \int d^4x V_6 \left(.... \right) =$$

$$= \int d^4x \left(M_{(4)}^2 \widetilde{R} + \frac{1}{4}\frac{1}{g_4^2} \widetilde{F}^2 \right) \quad . \qquad (10.96)$$

In the next subsections both cases shall be investigated, deriving the corresponding (quasilinear) differential equations in partial derivatives and moreover, finding concrete solutions of these equations for the special case of the metric (10.35) of a flat $4D$ Minkowski space, embedded in a five - dimensional ADS space of constant negative curvature. It will be shown also that for a definite scale factor $h(y) = \beta y^n$ (β is a constant) in front of the extra -coordinate y in the metric, the derived differential equations are still solvable, in spite of the fact that the five - dimensional space is no longer of a constant negative curvature. Besides the opportunity to extend the results to such spaces of non-constant curvature, there is one more reason for the necessity to investigate such quasilinear differential equations for concrete cases - in the next parts of this paper examples will be given, when such equations cannot be explicitely solved. **But evidently, some special kinds of metrics like (10.35) will allow the solution of these equations and consequently the determination of the scale length function $l(x)$ in terms of all the important parameters in the low - energy type I string thery action. If for certain metrics this is possible , then it will turn out to be possible to test whether there will be deviations from the standardly known gravitational theory with $l = 1$, if the electromagnetic coupling constant g_4, the $4D$ Planck constant $M_{(4)}$, the string scale m_s and the string coupling λ are known, presumably from future experiments or cosmological data**. Even if one assumes that there no deviations from the standard theory with $l = 1$, the obtained solutions will allow to find some new relations between the above mentioned parameters. It should be stressed that this refers

to the solutions of these equations, otherwise the obtained differential equations in the limit of $l = 1$ will result in the simple algebraic relations (10.91), already found in the literature.

One may also require the equivalence of the two approaches, expressed mathematically by (10.95) and (10.96), although for the moment it is not known whether there is some physical reason for this equivalence.

10.2.8. ALGEBRAIC RELATION AND A QUASILINEAR DIFFERENTIAL EQUATION IN PARTIAL DERIVATIVES FROM THE "RESCALED + COMPACTIFIED" LOW - ENERGY TYPE I STRING ACTION

In order to rewrite the "rescaled+compactified" string action (10.95), let us first define the "rescaled" square of the electromagnetic field strength as

$$\widetilde{F}^2 = \widetilde{F}_{AB}\widetilde{F}^{AB} = F_{AB}\widetilde{g}^{AM}\widetilde{g}^{BN}F_{MN} = \\ = l^2 F_{AB}g^{AM}g^{BN}F_{MN} = l^2 F^2 \quad . \tag{10.97}$$

Using the formulaes for the Riemann tensor and for the rescaled affine connection

$$\widetilde{\Gamma}^D_{AC} = \widetilde{g}^{DG}g_{GF}\Gamma^F_{AC} = l\Gamma^D_{AC} \quad , \tag{10.98}$$

the rescaled scalar gravitational curvature $\widetilde{R}$ can be written as

$$\widetilde{R} = \widetilde{g}^{DG}\widetilde{g}_{GF}\widetilde{R}_{ABCD} = \frac{1}{2}l^2 g^{AC}g^{BD}(g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - \\ -g_{BD,AC}) + l^4 g^{AC}g^{BD}g_{FG}(\Gamma^F_{CB}\Gamma^G_{AD} - \Gamma^F_{DB}\Gamma^G_{AC}) = \tag{10.99}$$

$$= l^4 R - \frac{1}{2}l^2(l^2-1)g^{AC}g^{BD}(g_{AD,BC} + g_{BC,AD} - g_{AC,BD} - g_{BD,AC}) \quad . \tag{10.100}$$

Substituting the above expressions (10.97) and (10.100) for $\widetilde{F}^2$ and $\widetilde{R}$ into the L. H. S. of the low - energy string action (10.95) and setting up equal the corresponding coefficients in front of the $\frac{1}{4}F^2$ term in the L. H. S. and the R. H. S. of (10.95), one can derive

$$\lambda = \frac{g_4 m_s^6 V_6}{(2\pi)^7}l^2 \quad . \tag{10.101}$$

This is almost the same expression as in (10.91), but now corrected wih the function of proportionality $l(\mathbf{x})$. The string coupling λ is thus a non - local physical quantity, depending on the space - time coordinates.

Next, after the elimination of the terms with $\frac{1}{4}F^2$ on both sides of (10.95) and substituting the found formulae for λ into the resulting expression on both sides of (10.95), one derives the algebraic relation

$$\left[\frac{(2\pi)^7}{V_6 m_s^4 g_4^4} - M^2_{(4)}\right] R = \frac{(2\pi)^7(l^2-1)}{2m_s^4 V_6 l^2 g_4^2}g^{AC}g^{BD}(....) \quad . \tag{10.102}$$

For brevity, the brackets (....) will denote the term in (10.100) with the second derivatives of the metric tensor. For $l=1$, as expected, we obtain the usual relation for $M^2_{(4)}$ as in (10.91). Therefore, physically any possible deviations from relation (10.91) can be attributed to the appearence of the new length scale $l(\mathbf{x})$. Let us introduce the notation

$$\beta \equiv \left[\frac{(2\pi)^7}{V_6 m_s^4 g_4^4} - M^2_{(4)}\right] m_s^4 V_6 \frac{2}{(2\pi)^7} \tag{10.103}$$

and assume that the deviation from the relation $M^2_{(4)} = \frac{(2\pi)^7}{V_6 m_s^4 g_4^2}$ is small, i.e. $\beta \ll 1$. Moreover, the number $(2\pi)^7$ in the denominator of (10.103) is great, so one can expect that β is really a small quantity, in spite of the fact that the string scale m_s (remember that in [47] it was set up ~ 1) and the compactification volume V_6 are not known. Then the lenght scale $l(x)$ can be expressed from the algebraic relation (10.102) as

$$l^2 = \frac{1}{1-\beta\frac{R}{g^{AC}g^{BD}(...)}} \approx 1 + \beta\frac{R}{g^{AC}g^{BD}(...)} \quad . \tag{10.104}$$

Consequently the deviation from the "standard" length scale $l=1$ in the case of a gravitational theory with $l \neq 1$ in the case of small β shall be proportional to the ratio $\frac{R}{g^{AC}g^{BD}(...)}$. In the concrete example of an $4D$ Minkowski space, embedded in a $5D$ ADS space of constant negative curvature, this ratio will be

$$\frac{R}{g^{AC}g^{BD}(...)} = \frac{(-8k^2)}{(-32k^2)} = \frac{1}{4} \tag{10.105}$$

and therefore, this constant factor will not affect the smallness of the number β.

Let us now derive the differential equation in partial derivatives, starting from the second representation of the "rescaled" scalar gravitational curvature $\widetilde{R}$ by means of the "rescaled" Ricci tensor $\widetilde{R}_{ij}$

$$\widetilde{R} = \widetilde{g}^{AB}\widetilde{R}_{AB} = l g^{AB}\left[\frac{\partial\widetilde{\Gamma}^C_{AB}}{\partial x^C} - \frac{\partial\widetilde{\Gamma}^C_{AC}}{\partial x^B} + \widetilde{\Gamma}^C_{AB}\widetilde{\Gamma}^D_{CD} - \widetilde{\Gamma}^D_{AC}\widetilde{\Gamma}^C_{BD}\right] \quad . \tag{10.106}$$

It can easily be found that the rescaled gravitational curvature is expressed through the usual one as

$$\begin{aligned}\widetilde{R} = \widetilde{g}^{AB}\widetilde{R}_{AB} = l^2 R + l^2(l-1)g^{AB}\left(\Gamma^C_{AB}\Gamma^D_{CD} - \Gamma^D_{AC}\Gamma^C_{BD}\right) + \\ + l\frac{\partial l}{\partial x^C} g^{AB}\Gamma^C_{AB} - l\frac{\partial l}{\partial x^B} g^{AB}\Gamma^C_{AC} \quad .\end{aligned} \tag{10.107}$$

Again, this expression and also (10.97) for $\widetilde{F}^2$ are substituted into the L. H. S. of the action (10.95) and the corresponding coefficients in front of the term $\frac{1}{4}F^2$ in the L. H. S. and the R. H. S. of (10.91) are set up equal. Thus one obtains

$$\lambda^2 = \frac{g_4^4 m_s^{12} V_6 l^4}{(2\pi)^{14}} \quad . \tag{10.108}$$

Substituting this expression into the resulting one on both sides of (10.91), we receive the following equation in partial derivatives in respect to the scale function $l(x)$:

$$\left[\frac{(2\pi)^7}{m_s^4 V_6 g_4^4 l^2} - M_4^2\right] R + \frac{(2\pi)^7}{m_s^4 V_6 g_4^4}\frac{(l-1)}{l^2} g^{AB}\left(\Gamma_{AB}^{C}\Gamma_{CD}^{D} - \Gamma_{AC}^{D}\Gamma_{BD}^{C}\right) +$$

$$+\frac{(2\pi)^7}{m_s^4 V_6 g_4^4}\frac{1}{l^3}\left[\frac{\partial l}{\partial x^C} g^{AB}\Gamma_{AB}^{C} - \frac{\partial l}{\partial x^B} g^{AB}\Gamma_{AC}^{C}\right] = 0 \quad . \qquad (10.109)$$

Note that for $l = 1$ (the known gravitational theory) we obtain again expression (10.91) for M_4^2. It turns out that for $l(x)$ we have both the algebraic relation (10.102) and the above differential equation (10.109).

Now there are two possibilities:

First possibility: The ten - dimensional space - time is represented as a factorized product of the compactification manifold V_6 and the remaining four - dimensional spacetime $K^{(4)}$, i.e. $V_6 \times K^{(4)}$. Then the coordinates of V_6 are independent from the coordinates of $K^{(4)}$ and consequently the derivatives of l, if calculated from the found algebraic relation (10.104), will depend not on the compactification volume, but on the ratio $\frac{R}{g^{AC}g^{BD}(...)}$:

$$\frac{\partial l}{\partial x^B} = \frac{\beta}{2} l^3 \frac{\partial}{\partial x^B}\left(\frac{R}{g^{AC}g^{BD}(...)}\right) \quad . \qquad (10.110)$$

If the above expression and also formulae (10.104) for l are substituted into the differential equation (10.109), then an algebraic relation in respect to V_6 is obtained, depending on the parameters in the initial string action (without the string coupling constant λ).

Second possibility: The ten - dimensional spacetime cannot be represented as factorized product and therefore the compactification volume V_6 depends on the coordinates of thee four - dimensional spacetime. Then an additional term $\frac{1}{2}\left(\frac{R}{g^{AC}g^{BD}(...)}\right) l^3 \frac{\partial \beta}{\partial x^B}$ has to be added to the derivative expression for $\frac{\partial l}{\partial x^B}$ in (10.110). Substituting into (10.109), a nonlinear differential equation in partial derivatives will be obtained in respect to the compactification volume.

10.2.9. (ANOTHER) ALGEBRAIC RELATION AND A QUASILINEAR DIFFERENTIAL EQUATION FROM THE "COMPACTIFIED+RESCALED" LOW ENERGY TYPE I STRING THEORY ACTION

This time we start from the action (10.96) and substitute the expressions for the "unrescaled" scalar quantities F^2 and R

$$F^2 = \frac{1}{l^2}\widetilde{F}^2 \quad ; \qquad R = \frac{1}{l^4}\widetilde{R} + \frac{(l^2-1)}{2l^2} g^{AC} g^{BD}(...) \qquad (10.111)$$

into the L. H. S. of (10.96).

Following the method, described in the previous section, we find for λ

$$\lambda = \frac{g_4 m_s^6 V_6}{(2\pi)^7 l^2} \quad , \tag{10.112}$$

which in respect to the function $l(x)$ can be considered as the "dual" one, if compared with (10.101). However, in comparison with (10.102), the obtained algebraic relation will be different

$$\left[\frac{(2\pi)^7}{V_6 m_s^4 g_4^4} - M_{(4)}^2\right] \widetilde{R} + \frac{(2\pi)^7 l^2 (l^2-1)}{2 m_s^4 V_6 g_4^4} g^{AC} g^{BD} (....) = 0 \quad . \tag{10.113}$$

If again expression (10.100) for $\widetilde{R}$ is used, the algebraic relation (10.113) can be rewritten as

$$\frac{1}{2} l^2 (l^2-1) g^{AC} g^{BD} (....) = \frac{P^2 R}{(P-NR)^2} + l^4 R \quad , \tag{10.114}$$

where P and N denote the expressions

$$P \equiv \frac{(2\pi)^7}{2 m_s^4 V_6 g_4^4} g^{AC} g^{BD} (...) \quad ; \quad N \equiv \frac{(2\pi)^7}{m_s^4 V_6 g_4^4} - M_4^2 \quad . \tag{10.115}$$

In the same way, starting from the second representation (10.107) of the gravitational Lagrangian for R in terms of $\widetilde{R}$ and again making use of formulae (10.107) for $\widetilde{R}$, one can obtain the **second quasilinear equation in partial derivatives**

$$\left[\frac{(2\pi)^7}{m_s^4 V_6 g_4^4} l^6 - M_4^2 l^4\right] R - \left[\frac{(2\pi)^7 l^2}{m_s^4 V_6 g_4^4} - M_4^2\right] \frac{l^2(l^2-1)}{2} g^{AC} g^{BD} (...) -$$

$$- \frac{(2\pi)^7 l^4 (l-1)}{m_s^4 V_6 g_4^4} g^{AB} \left(\Gamma_{AB}^C \Gamma_{CD}^D - \Gamma_{AC}^D \Gamma_{BD}^C\right) -$$

$$- \frac{(2\pi)^7 l^3}{m_s^4 V_6 g_4^4} \left(\frac{\partial l}{\partial x^C} g^{AB} \Gamma_{AB}^C - \frac{\partial l}{\partial x^B} g^{AB} \Gamma_{AC}^C\right) = 0 \quad . \tag{10.116}$$

Substituting the algebraic relation (10.114) into the second term of (10.116), the differential equation is obtained in a simpler form

$$\frac{(2\pi)^7 l^3}{m_s^4 V_6 g_4^4} \left(\frac{\partial l}{\partial x^C} g^{AB} \Gamma_{AB}^C - \frac{\partial l}{\partial x^B} g^{AB} \Gamma_{AC}^C\right) + \frac{R P^2}{l^3 (P-NR)^2} \left[\frac{(2\pi)^7 l^2}{m_s^4 V_6 g_4^4} - M_4^2\right] +$$

$$+ \frac{(2\pi)^7 l (l-1)}{m_s^4 V_6 g_4^4} g^{AB} \left(\Gamma_{AB}^C \Gamma_{CD}^D - \Gamma_{AC}^D \Gamma_{BD}^C\right) = 0 \quad . \tag{10.117}$$

This (second) differential equation evidently is different from the first one (10.109) and in this aspect an interesting conclusion can be made. Suppose that the two differential equations (10.109) and (10.117) simultaneously hold, which means that it does not matter whether we perform "rescaling + compactification" or "compactification +rescaling" in

the low energy type I string theory action. Then, if the initial term with the derivatives in (10.117) is expressed and substituted into the first differential equation (10.109), then the square of the length scale function l^2 can be found as a solution of the following algebraic equation

$$M_4^2 l^6 - \frac{(2\pi)^7}{m_s^4 V_6 g_4^4} l^4 + \frac{(2\pi)^7}{m_s^4 V_6 g_4^4} \frac{P^2}{(P - NR)^2} l^2 - M_4^2 \frac{P^2}{(P - NR)^2} = 0 \quad . \tag{10.118}$$

For the moment, this expression shall be used. Since the functions in the third and the fourth term depend on the function l, it would become clear later on, that the equation is a cubic one.

Both the above mentioned approaches of " rescaling + compactification " and "compactification + rescaling" would be consistent in the case of a non - imaginary Lobachevsky space, if the function $l(\mathbf{x})$ is a real one and not a complex one, i.e. the roots of the above equation should not be imaginary functions and there should be **at least** one root, which is a real function. This may lead additionallly to some restrictions on the parameters in the initial string action. Also, for $l = 1$ the above equation can be written as

$$\left[M_4^2 - \frac{(2\pi)^7}{m_s^4 V_6 g_4^4} \right] \left[\frac{P^2}{(P - NR)^2} + 1 \right] = 0 \quad . \tag{10.119}$$

There is no other relation from this equation besides the known one $M_4^2 - \frac{(2\pi)^7}{m_s^4 V_6 g_4^4} = 0$, since the nominator of the second term can be written as

$$2\left[\left(\frac{P}{N}\right)^2 - \left(\frac{P}{N}\right) R + \frac{1}{2} R^2 \right] = \frac{1}{2} \left[\left(\frac{P}{N} - \frac{R}{2}\right)^2 + \frac{R^2}{4} \right] \tag{10.120}$$

and evidently this term is positive and different from zero.

However, if solutions of the two quasilinear differential equations are found, then some new relations may be written. It should be kept in mind that these solutions are found by means of the characteristic system of equations, and the general solutions depend on the solutions of the characteristic system.

10.2.10. ALGEBRAIC INEQUALITIES FOR THE PARAMETERS IN THE LOW - ENERGY TYPE I STRING THEORY ACTION

Taking into account expressions (10.115) for P and N, eq.(10.118) after dividing by $Q^2 M_4^2 m_s^4 V_6 g_4^2$ can be written in the form of the following cubic algebraic equation in respect to the variable $l_1 = l^2$

$$l_1^3 + a_1 l_1^2 + a_2 l_1 + a_3 = 0 \quad , \tag{10.121}$$

where Q, a_1, a_2 and a_3 denote the expressions

$$Q \equiv \frac{g^{AC}g_{BD}(2\pi)^7 g_4^4(....)}{[g^{AC}g^{BD}(...)(2\pi)^7 g_4^4 - 2R\left((2\pi)^7 - M_4^2 V_6 m_s^4 g_4^4\right)]} \quad , \tag{10.122}$$

$$a_1 \equiv -\frac{(2\pi)^7}{M_4^2 m_s^4 V_6 g_4^2} \quad ; \quad a_2 \equiv \frac{(2\pi)^7}{M_4^2 m_s^4 V_6 g_4^2 Q^2} \quad ; \quad a_3 \equiv -\frac{g_4^2}{Q^2} \quad . \tag{10.123}$$

After the variable change $l_1 = x - \frac{a_1}{3}$ equation (10.121) is brought to the form

$$x^3 + ax + b = 0 \quad , \tag{10.124}$$

where a and b are the expressions

$$a \equiv a_2 - \frac{a_1^2}{3} \quad ; \quad b \equiv 2\frac{a_1^3}{27} - \frac{a_1 a_2}{3} + a_3 \quad . \tag{10.125}$$

The roots of the cubic equation (10.124) are given by the formulae [9]

$$x = \sqrt[3]{p} - \frac{a}{3\sqrt[3]{p}} \quad , \tag{10.126}$$

where p denotes the expression

$$p \equiv -\frac{b}{2} \pm \sqrt{\frac{b^2}{4} + \frac{a^3}{27}} \quad . \tag{10.127}$$

The roots of the cubic equation will not depend on the $+$ or $-$ sign in front of the square in the above expression.

It may be noted that if the expression for $\frac{b^2}{4} + \frac{a^3}{27}$ is negative, then the corresponding roots x_1, x_2, x_3 and the length function $l(x)$ will be imaginary. From a physical point of view, this would be unacceptable, but with one exception - in the imaginary Lobachevsky space [73], which is realized by all the straight lines outside the absolute cone (on which the scalar product is zero, i.e. $[x, x] = 0$), the length may may take imaginary values in the interval $[0, \frac{\pi\, i}{2k}]$ (k is the Lobachevsky constant). Further we shall assume that $l(x)$ is a real function, but in principle it is interesting that the sign of the inequalities, relating the paeameters in the string action, may change, if the spacetime is an imaginary Lobachevsky one.

The expression for $\frac{b^2}{4} + \frac{a^3}{27}$ can be written as

$$\frac{b^2}{4} + \frac{a^3}{27} = \frac{1}{Q^6 d^6}[\frac{1}{27}g_4^6 d^3 - \frac{1}{4.27}g_4^8 d^2 Q^2 - \frac{2}{27^2}g_4^{12}Q^6 +$$

$$+\frac{1}{4}g_4^2 Q^2 - \frac{1}{6}g_4^6 Q^2 d^4 + \frac{1}{27}g_4^8 Q^4 d^3] \quad , \tag{10.128}$$

where d is the introduced notation for

$$d \equiv \frac{M_4^2 V_6 m_s^4 g_4^4}{(2\pi)^7} \quad . \tag{10.129}$$

It is difficult to check when expression (10.128) will be non - negative, since Q depends also on d and a higher - degree polynomial in respect to d will be obtained. However, it may be noted that since

$$l^2 = l_1 = x - \frac{a_1}{3} > 0 \tag{10.130}$$

and since a_1 is a non - complex quantity, then all the roots x_1, x_2, x_3 are real. Therefore from the Wiet formulae

$$-a = x_1 + x_2 + x_3 > a_1 \tag{10.131}$$

and with account of the expressions for a and a_1 an equality can be obtained in respect to d

$$\frac{1}{3}g_4^2 > W(W+2)d^4 - 2W(W+1)d^3 + W^2 d^2 \qquad , \tag{10.132}$$

where W is the notation for

$$W \equiv \frac{2R}{g^{AB}g^{CD}(...)g_4^4} \qquad . \tag{10.133}$$

The last (third) inequality with the parameters of the low - energy type I string theory action can be obtained from the restriction (10.130) $x > \frac{a_1}{3}$ for the roots of the cubic equation and expression (10.126) for x

$$\frac{3\sqrt[3]{p^2} - a}{3\sqrt[3]{p}} > \frac{a_1}{3} \qquad . \tag{10.134}$$

Denoting

$$q_1 = \sqrt[3]{p^2} \quad , \tag{10.135}$$

the above inequality can be rewritten as

$$9q_1^2 - (a_1^2 + 6a)q_1 + a^2 > 0 \qquad , \tag{10.136}$$

which is satisfied for

$$p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b\sqrt{\frac{b^2}{2} + \frac{a^3}{27}} > \left[\frac{a_1 + 6a}{18} + \frac{a_1}{18}\sqrt{a_1^2 + 12a}\right]^3 \tag{10.137}$$

or for

$$p^2 = \frac{b^2}{2} + \frac{a^3}{27} - b\sqrt{\frac{b^2}{2} + \frac{a^3}{27}} < \left[\frac{a_1 + 6a}{18} - \frac{a_1}{18}\sqrt{a_1^2 + 12a}\right]^3 \qquad . \tag{10.138}$$

10.2.11 SOLUTIONS OF THE FIRST QUASILINEAR DIFFERENTIAL EQUATION (10.109) FOR THE CASE OF A FLAT $4D$ MINKOWSKI METRIC, EMBEDDED IN A FIVE - DIMENSIONAL SPACE (OF CONSTANT NEGATIVE OR NON - CONSTANT CURVATURE)

The purpose will be to show that the differential equation in partial derivatives (10.109) will be solvable for the previously considered case of the metric (10.12), written now as

$$ds^2 = e^{-2k\epsilon y}\eta_{\mu\nu}dx^\mu dx^\nu + h(y)dy^2 \qquad , \tag{10.139}$$

$\eta_{\mu\nu} = (+,-,-,-)$ with $h(y) = 1$ and $\epsilon = \pm 1$. Moreover, the equation will be solvable also for the case of a power - like dependence of the scale factor $h(y) = \gamma y^n$ ($\gamma = const$), when the five - dimensional scalar curvature is no longer a constant one. In principle, it is necessary to know for what kind of metrics quasilinear differential equations of the type (10.109) admit exact analytical solutions, since it may be shown that for more complicated metrics (for example, when the embedded four - dimensional spacetime is a Schwarzschild Black hole with an warp factor), such analytical solutions cannot be found in the sense that some algebraic relations can be found but the unknown function cannot be expressed from them. This will be shown also in the next parts of this paper.

As usual, the Greek indices μ, ν, α, β will run from 1 to 4 and the extra - dimensional metric tensor component is $h(y) \equiv g_{55}$. The big letters $A, B, C...$ will denote the coordinates of the five - dimensional spacetime.

The corresponding affine connection components are

$$\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{5\nu} = 0 \quad ; \quad \Gamma^{5}_{\mu\nu} = \frac{k\varepsilon}{h}\eta_{\mu\nu}e^{-2k\varepsilon y} \quad , \tag{10.140}$$

$$\Gamma^{5}_{\mu 5} = 0 \;\; ; \;\; \Gamma^{5}_{55} = \frac{1}{2}\frac{h^{'}(y)}{h(y)} \quad ; \quad \Gamma^{\alpha}_{55} = \frac{1}{2}e^{2k\varepsilon y}\eta_{\alpha\alpha}h^{'}(y) \quad . \tag{10.141}$$

The expressions for the scalar curvatureb$R$ and for $g^{AB}(\Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B})$ are

$$R = -\frac{8k^2}{h} - 4k\epsilon\frac{h^{'}}{h^2} \;\; = g^{AB}(\Gamma^{C}_{AB,C} - \Gamma^{C}_{AC,B}) \;\; , \tag{10.142}$$

from where, taking the difference of the two expressions, it is found that $g^{AB}(\Gamma^{C}_{AB}\Gamma^{D}_{CD} - \Gamma^{D}_{AC}\Gamma^{C}_{BD}) = 0$ and the differential equation (10.109) is written as

$$\frac{(2\pi)^7}{m_s^4 V_5 g_5^5}\frac{e^{2k\epsilon y}h^{'}}{2h}\left[\frac{\partial l}{\partial x^1} - \frac{\partial l}{\partial x^2} - \frac{\partial l}{\partial x^3} - \frac{\partial l}{\partial x^4}\right] +$$

$$+\frac{(2\pi)^7 4k\epsilon}{m_s^4 V_5 g_5^4 h}\frac{\partial l}{\partial y} = Cl - Dl^3 \;\; , \tag{10.143}$$

where C and D denote the expressions

$$C \equiv \frac{(2\pi)^7 4k(2kh + \epsilon h^{'})}{m_s^4 V_5 g_5^4 h^2} \quad ; \quad D \equiv \frac{M_5^2 4k(2kh + \epsilon h^{'})}{h^2} \quad . \tag{10.144}$$

The characteristic system of equations for the equation (10.143) is

$$\frac{dl}{Cl - Dl^3} = \frac{\varepsilon m_s^4 V_5 g_5^4 h}{(2\pi)^7 4k}dy = d\sigma \quad , \tag{10.145}$$

$$\frac{2m_s^4 V_5 g_5^4 e^{-2k\varepsilon y}}{(2\pi)^7(\ln h)^{'}}dx^1 = -\frac{2m_s^4 V_5 g_5^4 e^{-2k\varepsilon y}}{(2\pi)^7(\ln h)^{'}}dx^i = d\sigma \quad , \tag{10.146}$$

where the indice $i = 2, 3, 4$ and σ is some parameter. The solution of the first characteristic equation for the y and l variables is

$$\frac{\varepsilon_1 l}{[\varepsilon_2(l^2 - \alpha_1^2)]^{\frac{1}{2}}} = C_1(x_1, x_i) e^{2k\varepsilon_3 y} h \qquad , \tag{10.147}$$

where

$$\alpha_1 = \sqrt{\frac{C}{D}} \tag{10.148}$$

and ε_1, ε_2, ε_3 take values ± 1 independently one from another.

In order to find the function $C_1(x_1, x_i)$, the obtained solution (10.147) should be differentiated by x_1 and the characteristic equations for $\frac{\partial l}{\partial y}$ and $\frac{\partial y}{\partial x_1}$ have to be used. As a result, the function $C_1(x_1, x_i)$ is found as a solution of the following simple differential equation

$$M = S \frac{\partial C_1(x_1, x_i)}{\partial x_1} + T C_1(x_1, x_i) \qquad , \tag{10.149}$$

where the functions M, S and T are defined as follows

$$M \equiv - \frac{\varepsilon_1 (2l^2 - \alpha_1^2) l 8k M_5^2 (2kh + \varepsilon_3 h^{'}) m_s^4 V_5 g_5^4}{[\varepsilon_2(l^2 - \alpha_1^2)]^{\frac{1}{2}} (2\pi)^7 h h^{'} e^{2k\varepsilon_3 y}} \quad \text{for} \quad \varepsilon_1 \varepsilon_2 = -1 \quad , \tag{10.150}$$

$$M \equiv \frac{\varepsilon_1 l \alpha_1^2 8k M_5^2 (2kh + \varepsilon_3 h^{'}) m_s^4 V_5 g_5^4}{[\varepsilon_2(l^2 - \alpha_1^2)]^{\frac{1}{2}} (2\pi)^7 h h^{'} e^{2k\varepsilon_3 y}} \quad \text{for} \quad \varepsilon_1 \varepsilon_2 = +1 \qquad , \tag{10.151}$$

$$S \equiv h e^{2k\varepsilon_3 y} \quad ; \quad T \equiv \frac{8k(2k + \varepsilon_3 h^{'})}{h^{'}} \qquad . \tag{10.152}$$

The solution of the differential equation (10.149) can be written as

$$C_1(x_1, x_i) = \frac{M}{T} - \varepsilon_4 \frac{C_2(x_i)}{T} e^{-\int \frac{T}{S} dx_1} \qquad . \tag{10.153}$$

Again, the obtained solution (10. 147) can be differentiated in respect to x_i and taking into account from the characteristic equations that

$$\frac{\partial l}{\partial x_i} = - \frac{\partial l}{\partial x_1} \quad ; \quad \frac{\partial y}{\partial x_i} = - \frac{\partial y}{\partial x_1} \qquad , \tag{10.154}$$

the solution of the corresponding equation (10.149) with $\widetilde{M} = -M$, $\widetilde{T} = -T$, $\widetilde{S} = S$ can be represented as

$$C_1(x_1, x_i) = \frac{M}{T} + \varepsilon_4 \frac{C_3(x_1)}{T} e^{\int \frac{T}{S} dx_i} \qquad . \tag{10.155}$$

Substracting the two relations (10.153) and (10.155), the following relation between the functions $C_2(x_i)$ and $C_3(x_1)$ can be derived

$$C_2(x_i) = e^{\int \frac{T}{S} dx_1} C_3(x_1) e^{\int \frac{T}{S} dx_i} \qquad . \tag{10.156}$$

Now let us differentiate relation (10.155) in respect to x_1 and make use of (10.156) and its derivative in respect to x_1. Then the following differential equation is derived in respect to the function $C_3(x_1)$

$$\frac{\partial C_3(x_1)}{\partial x_1} - C_3(x_1)\frac{T}{S} = 0 \quad , \tag{10.157}$$

from where

$$C_3(x_1) = const.\ e^{\int \frac{T}{S} dx_1} \quad . \tag{10.158}$$

Therefore from (10.155)

$$C_1(x_1, x_i) = \frac{M + \varepsilon_4}{T} \tag{10.159}$$

and substituting into (10.147), a final expression for l can be found (for the case $\varepsilon_1\varepsilon_2 = +1$)

$$l^2 = \frac{\varepsilon_2 \alpha_1^2 e^{4\varepsilon_3 k y} h^2}{\varepsilon_2 h^2 e^{4\varepsilon_3 k y} - \frac{T^2}{(M+\varepsilon_4)^2}} \quad . \tag{10.160}$$

The solution for the other case $\varepsilon_1\varepsilon_2 = -1$ can be found analogously.

The general solution of the quasilinear differential equation in partial derivativatives will be not only expression (10.160), but also any function V, depending on the first integrals K_1, K_2, K_3,....,K_6 of the characteristic system of equations [74]

$$V = V(K_1, K_2, K_3, K_4, K_5, K_6) \quad . \tag{10.161}$$

Now let us find a solution of the characteristic equation for the x_i and y variables for the case of the function $h(y) = \gamma y^n$. The equation can be written as

$$e^{2k\varepsilon_4 y}\ y^{n-1} = -\varepsilon_4 \frac{8k}{n\gamma} dx^i \quad , \tag{10.162}$$

from where x^i can be expressed as

$$x^i = -\varepsilon_4 \frac{n\gamma}{8k} I_{n-1}(k, y) \tag{10.163}$$

and $I_{n-1}(k, y)$ denotes the integral

$$I_{n-1}(k, y) = \int e^{2k\varepsilon_4 y} y^{n-1} dy \quad . \tag{10.164}$$

This integral can be exactly calculated (see Appendix D). Note that because of the complicated expression for the integral $I_{n-1}(k, y)$, y cannot be expressed as a function of the x_i and the x_1 coordinate. Also, the solvability of the quasilinear differential equation is determined mostly by the presence of the embedded flat $4D$ Minkowski spacetime. Therefore, it may be expected that there might be another functions $h(y)$, for which exact analytical solution may be found.

10.2.12 SOLUTIONS OF THE SECOND QUASILINEAR DIFFERENTIAL EQUATION (10.116) FOR THE CASE OF A FLAT 4*D* MINKOWSKI METRIC, EMBEDDED IN A FIVE - DIMENSIONAL SPACE

The same approach, developed in the previous subsection, shall be applied in respect to the second quasilinear differential equation in partial derivatives (10.116). The aim will be to show that the analytical solution will be different, compared to the first one for the differential equation (10.109).

The differential equation (10.116) for the case of the metric (10.139) can be written as

$$-D\frac{\partial l}{\partial x^1}+D\left[\frac{\partial l}{\partial x^2}+\frac{\partial l}{\partial x^3}+\frac{\partial l}{\partial x^4}\right]+E\frac{\partial l}{\partial y}+Al^4+Bl^2+C=0 \quad , \tag{10.165}$$

where A,B,C,D,E denote the expressions

$$A\equiv\frac{(2\pi)^7 4k(2kh-\varepsilon h^{'})}{h^2 m_s^4 V_5 g_s^4} \quad ; \quad C\equiv\frac{16k^2M_5^2}{h} \quad , \tag{10.166}$$

$$B\equiv\frac{16k^2(2\pi)^7}{hm_s^4 g_5^4 V_5}+\frac{4kM_5^2}{h^2}(\varepsilon h^{'}-2kh) \quad , \tag{10.167}$$

$$D\equiv-\frac{(2\pi)^7 l e^{2k\varepsilon y}h^{'}}{m_s^4 V_5 g_5^4 2h} \quad ; \quad E\equiv-\frac{4k\varepsilon(2\pi)^7 l}{m_s^4 V_5 g_5^4 h} \quad . \tag{10.168}$$

The characteristic system of equations is

$$\frac{dx_1}{D}=-\frac{dx_i}{D}=-\frac{dy}{E}=\frac{dl}{Al^4+Bl^2+C} \quad . \tag{10.169}$$

The characteristic equation for the y and l variables can be written as

$$d\left[\ln\left(\varepsilon_1\left(\frac{s-G}{s+G}\right)\right)\right]=2AG\frac{\varepsilon_4 h m_s^4 V_5 g_5^4}{4k(2\pi)^7}dy \quad , \tag{10.170}$$

where

$$G\equiv\frac{B^2}{4A^2}-\frac{C}{A}=\frac{M_5\sqrt{(F-\frac{k}{2})^2+k^2\left(2(2\pi)^7-\frac{1}{4}\right)}}{\sqrt{\pi}(2\pi)^3\mid 2k-\varepsilon_4\frac{h^{'}}{h}\mid} \tag{10.171}$$

is a **non - negative function** and

$$s\equiv l^2+\frac{B}{2A} \quad ; \quad F\equiv m_s^4 V_5 g_5^4(2k-\varepsilon_4\frac{h^{'}}{h}) \quad . \tag{10.172}$$

The integration of the characteristic equation (10.170) results in the expression

$$l^2=-\frac{B}{2A}+G\frac{(1+\varepsilon_1 D_1(x_1,x_i)e^{Z\ \widetilde{J}(y)})}{(1-\varepsilon_1 D_1(x_1,x_i)e^{Z\ \widetilde{J}(y)}} \quad , \tag{10.173}$$

where Z is the expression

$$Z \equiv \frac{2M_5 \varepsilon_4 m_s^2 g_5^2 \sqrt{V_5}}{(2\pi)^3} \tag{10.174}$$

and $\widetilde{J}(y)$ is the integral

$$\widetilde{J}(y) \equiv \int \frac{\sqrt{K_1 y^2 + K_2 y + 1}}{y} dy \quad . \tag{10.175}$$

The functions K_1 and K_2 are the following

$$K_1 \equiv \frac{k^2 \left[2(2\pi)^7 - \frac{1}{4}\right]}{m_s^8 V_5^2 g_5^8} + k^2 \left(2 - \frac{1}{2m_s^4 V_5 g_5^4}\right)^2 \quad , \tag{10.176}$$

$$K_2 \equiv -2\varepsilon_4 \left(2k - \frac{k}{2m_s^4 V_5 g_5^4}\right) \quad . \tag{10.177}$$

It is important to note that the free term under the square in the integral $\widetilde{J}(y)$ (10.175) is positive (it is $+1$) and the function K_1 is also positive. The analytical solution of integrals of the type (10.175) depends on the sign of the function K_1 and of the free term, which in the present case are both positive. The explicite solution can be found in the book of Timofeev [75]:

$$\widetilde{J}(y) \equiv \sqrt{K_1 y^2 + K_2 y + 1} + \frac{K_2}{2\sqrt{K_1}} \ln(K_1 y + \frac{K_2}{2} +$$

$$+ \sqrt{K_1}\sqrt{K_1 y^2 + K_2 y + 1}) - \ln\left(\frac{1 + \frac{K_2}{2} y + \sqrt{K_1 y^2 + K_2 y + 1}}{y}\right) \quad . \tag{10.178}$$

Now it remains to determine the function $D_1(x_1, x_i)$ in expression (10.173) for l^2. For the purpose, let us denote the under - integral expression in (10.175) by $J(y)$, differentiate both sides of (10.173) by x_1 and take into account the expressions for $\frac{\partial l}{\partial x_1}$ and $\frac{\partial y}{\partial x_1}$ from the characteristic system of equations. After rearranging the terms and denoting by V

$$V \equiv \varepsilon_1 \frac{-\frac{2l(Al^4 + Bl^2 + C)}{E} + \frac{\partial}{\partial y}\left(\frac{B}{2A}\right) - \frac{\partial Q}{\partial y}\frac{(l^2 + \frac{B}{2A})}{G}}{G + l^2 + \frac{B}{2A}} \quad , \tag{10.179}$$

the following differential equation can be obtained for the function $D_1(y)$

$$\frac{\partial D_1}{\partial y} + D_1(ZJ(y) + V) - Ve^{-Z\widetilde{J}(y)} = 0 \quad . \tag{10.180}$$

It may seem strange at first glance that the function D_1 depends on the y coordinate, while in (10.173) it was assumed that $D_1 = D_1(x_1, x_i)$. In fact, from the characteristic equations (10.169) for the y and x_1 variables it follows

$$\frac{dy}{e^{2k\varepsilon_4 y} h'} = -\frac{\varepsilon_4}{8k} dX_1 \quad . \tag{10.181a}$$

For the concrete expression for the function $h(y)=\gamma y^n$, the coordinates x_1 and x_i can be expressed as

$$x_1=-\varepsilon_4\frac{8k}{n\gamma}I(-k,1-n)+const.\ \ ;\ \ x_i=-x_1 \tag{10.181b}$$

and therefore it is reasonable to consider that $D_1=D_1(x_1(y),x_i(y))=D_1(y)$. Note however that due to the complicated structure of the integral $I(-k,1-n)$, it is impossible to express y as a function of x_1 (or x_i).

As in the previous case, the general solution of the equation depends on all the first integrals of the characteristic system of equations.

10.2.13 LENGTH FUNCTION $l(x)$ FROM THE CONSTANCY OF THE SCALAR CURVATURE R UNDER "RESCALINGS" OF THE CONTRAVARIANT METRIC TENSOR FOR THE CASE OF A FLAT $4D$ MINKOWSKI METRIC, EMBEDDED IN A $5D$ SPACETIME.

Now we shall find solutions of the corresponding differential equation in partial derivatives, when the second representation of the scalar curvature $\widetilde{R}$ (10.107) is equal to the initial scalar curvature R (i.e. $\widetilde{R}=R$). The obtained differential equation under this identification is

$$l^3\left[R-\frac{1}{2}g^{AC}g^{BD}(...)\right]+l^2\left[-R+g^{AB}(\Gamma^C_{AB,C}-\Gamma^C_{AC,B})\right]+$$
$$+l\left[\frac{1}{2}g^{AC}g^{BD}(...)-g^{AB}(\Gamma^C_{AB,C}-\Gamma^C_{AC,B})\right]+$$
$$+\frac{\partial l}{\partial x^B}g^{AB}\Gamma^C_{AC}-\frac{\partial l}{\partial x^C}g^{AB}\Gamma^C_{AB}=0 \quad . \tag{10.182.}$$

The expression in the small brackets is the same as in (10.99), i.e. $(...)\equiv(g_{AD,BC}+g_{BC,AD}-g_{AC,BD}-g_{BD,AC})$. The equation (10.182) for the case of the metric (10.107) with the affine connection components (10.139) acquires the form

$$\varepsilon\frac{\partial l}{\partial y}+\frac{h^{'}}{8k}e^{2k\varepsilon y}\left(\frac{\partial l}{\partial x_1}-\frac{\partial l}{\partial x_2}-\frac{\partial l}{\partial x_3}-\frac{\partial l}{\partial x_4}\right)=$$
$$=(2k-\varepsilon\frac{h^{'}}{h})(l^3-l) \quad . \tag{10.183.}$$

The characteristic system of equations is

$$\frac{dl}{(2k-\varepsilon\frac{h^{'}}{h})l(l^2-1)}=\varepsilon dy= \tag{10.184.}$$

$$=\frac{dx_1}{h^{'}}8ke^{-2k\varepsilon y}=-\frac{dx_i}{h^{'}}8ke^{-2k\varepsilon y} \quad . \tag{10.185.}$$

The solutions of the characteristic system for the x_1 and y variables are correspondingly

$$x_1 = C_4(x_i, l) + \varepsilon_2 \frac{e^{2k\varepsilon_1 y} h}{8k} - \frac{1}{4} \int h e^{2k\varepsilon_1 y} dy \quad , \tag{10.186.}$$

$$l^2 = \frac{h^2}{h^2 - D_2(x_1, x_i) e^{4k\varepsilon_1 y}} \quad . \tag{10.187.}$$

Unfortunately, (10.187) cannot be considered as an expression for l, since from (10.186) it is obvious that $D_2(x_1, x_i)$ also depends on the function l. Also, it should be understood that D_2 depends on all the variables x_i, $i = 2, 3, 4$.

Let us differentiate both sides of (10.187) by x_1

$$2l \frac{\partial l}{\partial x_1} = \frac{2lh^{'}}{h} \frac{\partial y}{\partial x_1} - \frac{l^4}{h^2} (2hh^{'} \frac{\partial y}{\partial x_1} -$$

$$-2D_2 \frac{\partial D_2}{\partial x_1} e^{4k\varepsilon_1 y} - D_2 4k\varepsilon_1 e^{4k\varepsilon_1 y} \frac{\partial y}{\partial x_1}) \quad . \tag{10.188.}$$

If the same operation is applied also in respect to the x_i coordinate and the derived equation is summed up with (10.188) with account also of $\frac{\partial l}{\partial x_1} = -\frac{\partial l}{\partial x_i}$, $\frac{\partial y}{\partial x_1} = -\frac{\partial y}{\partial x_i}$, then it can be obtained

$$\frac{2l^4}{h^2} e^{4k\varepsilon_1 y} D_2 \left(\frac{\partial D_2}{\partial x_1} + \frac{\partial D_2}{\partial x_i} \right) = 0 \quad . \tag{10.189.}$$

The equality is satisfied also for $D_2 \equiv 0$, which evidently corresponds to the standard case $l = 1$ in gravity theory. The other case, when the equality is fulfilled, is $\frac{\partial D_2}{\partial x_1} = -\frac{\partial D_2}{\partial x_i}$.

Now let us rewrite equation (10.188) with account of the expressions for $\frac{\partial l}{\partial x_1}$ and $\frac{\partial y}{\partial x_1}$ from the characteristic system of equations (10.184 - 10.185). Then the following nonlinear differential equation in respect to the function D_2^2 is derived

$$\frac{\partial D_2^2}{\partial y} - 4k\varepsilon_1 (1 + \frac{4}{h}) D_2^2 -$$

$$-2\varepsilon_1 \varepsilon_5 \frac{h^{'}}{h} (2k - \varepsilon_1 \frac{h^{'}}{h}) e^{-4k\varepsilon_1 y} (h^2 - D_2^2 e^{4k\varepsilon_1 y})^{\frac{3}{2}} = 0 \quad . \tag{10.190.}$$

Finding the function $D_2 = D_2(y)$ as a solution of this equation, from (10.187) l^2 can also be found as a function of the extra - coordinate y, i.e. $l = l(y)$. But then, differentiating the solution for x_1 (10.186) by y, the function $C_4(x_i, l)$ can be found as a solution of the following differential equation

$$\frac{\partial C_4(x_i, l)}{\partial y} = F_1 \quad , \tag{10.191.}$$

where the function F_1 is determined as

$$F_1 \equiv e^{2k\varepsilon_1 y} [\frac{\varepsilon_2 l (l^2 - 1) h^{'}}{8k} - \frac{\varepsilon_1 \varepsilon_2 h}{4} - \frac{\varepsilon_2 h^{'}}{8k} + \frac{h}{4}] \tag{10.192.}$$

and l has to be substituted with expression (10.187), in which D_2 is determined as a solution of the differential equation (10.190). The representation in the form (10.191) is particularly convenient for the case $h(y) = \gamma y^n$, when the difficulty will be only in calculaing the integral along y in the first term of (10.192). The advantage of the representation (10.191) will become evident if we differentiate the solution (10.186) by l, obtaining thus the differential equation

$$E_1(y) = \frac{\partial C_4(x_i, l)}{\partial l} + \frac{E_2(y)}{l(l^2 - 1)} \quad , \qquad (10.193.)$$

where

$$E_1(y) \equiv \frac{hh^{'} e^{2k\varepsilon_1 y}}{8k(2kh - \varepsilon_2 h^{'})} \quad , \qquad (10.194.)$$

$$E_2(y) \equiv \frac{\varepsilon_2 h e^{2k\varepsilon_1 y}}{(2kh - \varepsilon_2 h^{'}} \left[(\varepsilon_1 \varepsilon_2 - 1) + \frac{\varepsilon_2 h^{'}}{8k} \right] \quad . \qquad (10.195.)$$

Now it is important to stress that the second representation (10.193) is **inconvenient** to use for the case $h(y) = \gamma y^n$. The reason is that the integration is along the l coordinate, which means that the y coordinate in $E_1(y)$ and $E_2(y)$ has to be expressed from expression (10.187) as a function of l. However, in view of the extremely complicated expression, this is not possible. Instead, the differential equation (10.193) will be very helpful for the case $h(y) \equiv 1$, which is frequently encountered in most of the papers on theories with extra dimensions. Indeed, then the nonlinear differential equation (10.190) is of a particularly simple form:

$$\frac{\partial D_2^2}{\partial y} - 20k\varepsilon D_2^2 = 0 \quad , \qquad (10.196.)$$

from where with the help of (10.187)

$$l^2 = \frac{1}{1 - const.e^{24k\varepsilon_1 y}} \quad . \qquad (10.197.)$$

Note one interesting property of the obtained solution, already mentioned in the Introduction - **when $\varepsilon_1 = -1$ and y tends to infinity, the known case in gravity theory $l^2 = 1$ is recovered.** Since we received this solution for a partial case, it would be interesting to check whether this happens also for an arbitrary function $h(y)$. But this is more complicated since it is necessary to find the solution of the nonlinear differential equation (10.190).

Now y can be expressed easily and the resulting differential equation (10.193) with $E_1(y) = 0$ can be rewritten as

$$\frac{\partial C_4(x_i, l)}{\partial l} + \frac{\varepsilon_2 (\varepsilon_1 \varepsilon_2 - 1) e^{\frac{1}{12}}}{const.8kl^3} = 0 \quad . \qquad (10.198.)$$

The solution of the equation can be represented as

$$C_4(x_i, l) = -\frac{\varepsilon_2 e^{\frac{1}{12}} (1 - const\ e^{24k\varepsilon_1 y})}{8k\ const.} \widetilde{C}_4(x_i) \quad . \qquad (10.199.)$$

The unknown function $\widetilde{C}_4(x_i)$ can be found if expression (10.186) for x_1 is differentiated in respect to x_i. Unfortunately, the resulting formulae will contain the expressions for $\frac{\partial y}{\partial x_i}$ and $\frac{\partial l}{\partial x_i}$, which are singular when $h^{'}(y) = 0$.But if we multiply by $\frac{\partial x_i}{\partial y}$, the following differential equation will be obtained

$$\frac{\partial x_1}{\partial y} = \frac{\varepsilon_1(\varepsilon_2 - \varepsilon_1)}{4} e^{2k\varepsilon_1 y} + \frac{\varepsilon_2 e^{\frac{1}{12}}}{8k \ const \ l^3} \widetilde{C}_4(x_i) \frac{\partial l}{\partial y} -$$

$$- \frac{\varepsilon_2 e^{\frac{1}{12}}}{8k \ const \ l^2} \frac{\partial \widetilde{C}_4(x_i)}{\partial y} \quad , \tag{10.200}$$

which is no longer singular in the limit $h^{'}(y) = 0$, because the expressions for $\frac{\partial x_1}{\partial y}$ and $\frac{\partial l}{\partial y}$ are

$$\frac{\partial x_1}{\partial y} = 0 \quad ; \quad \frac{\partial l}{\partial y} = \varepsilon_2 2kl(l^2 - 1) \quad . \tag{10.201}$$

Taking into account these formulaes, the solution of the differential equation (10.200) in respect to the function $\widetilde{C}_4(x_i)$ can be found in the form

$$\widetilde{C}_4(x_i) = const_2 \mid 1 - const \ e^{24k\varepsilon_1 y} \mid^{-\frac{\varepsilon_1}{12}} \quad . \tag{10.202}$$

Substituting into (10.199), the final expressions for the function $C_4(x_i, l)$ can be found and also for the coordinate x_1, which for $\varepsilon_1 = \varepsilon_2$ and $h(y) = 1$ is simply

$$x_1 = C_4(x_i(y), l(y)) =$$

$$= - \frac{\varepsilon_2 \ const_2 \ e^{\frac{1}{12}}}{8k \ const} \frac{\left(1 - const \ e^{24k\varepsilon_1 y}\right)}{\mid 1 - const \ e^{24k\varepsilon_1 y} \mid^{\frac{\varepsilon_1}{12}}} \quad . \tag{10.203}$$

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11 APPENDIX A: SOME PROPERTIES OF THE NEWLY INTRODUCED CONNECTION $\widetilde{\Gamma}_{ij}^{k} = \frac{1}{2}dX^{k}dX^{l}(g_{jl,i} + g_{il,j} - g_{ij,l})$.

A1: FIRST AND SECOND DIFFERENTIALS

The first and the second differentials $dX^k = dX^k(x^1, x^2, ..., x^n)$ and $d^2X^i = d^2X^k(x^1, x^2, ..., x^n)$ are functions, in principle, of the initial coordinates $x^1, x^2, ..., x^n$. However, it is important to note that since the mapping $X^i = X^i(x^1, x^2, ..., x^n)$ is unique and therefore det $\parallel \frac{\partial X^k}{\partial x^l} \parallel \neq 0$, the inverse mapping $x^k = x^k(X^1, X^2, ..., X^n)$ can also be defined. Therefore we may also write

$$d^2X^i = d^2X^i(x^1, x^2, ..., x^n) = d^2X^i(X^1, X^2, ..., X^n) = \tag{A1}$$

$$= \sum_{r=1}^{n} \frac{\partial(dX^k)}{\partial X^r} dX^r \quad . \tag{A2}$$

Our next purpose will be to express the second differential, starting from the R. H. S. of the above expression, through the initial coordinates and thus really to prove that the second differential is one and the same in all system of coordinates. Let us transform the R. H. S. of (A1) in the following way

$$d^2X^i(X^1, X^2, ..., X^n) = \sum_{s=1}^{n} \frac{\partial}{\partial X^s} \left[\sum_{r=1}^{n} \frac{\partial(dX^k)}{\partial X^r} dX^r \right] \sum_{f=1}^{n} \frac{\partial X^s}{\partial x^t} dx^t = \tag{A3}$$

$$= \sum_{r,s,f=1}^{n} \frac{\partial}{\partial x^p} \left[\frac{\partial X^k}{\partial x^r} dx^r \right] \frac{\partial x^p}{\partial X^s} \frac{\partial X^s}{\partial x^t} dx^t = \tag{A4}$$

$$= \sum_{r,t} \frac{\partial^2 X^k}{\partial x^t \partial x^r} dx^r dx^t + \sum_{r} \frac{\partial X^k}{\partial x^r} d^2x^r \quad , \tag{A5}$$

where due to the existence of the inverse transformation $x^k = x^k(X^1, X^2, ..., X^n)$ it has been used that

$$\frac{\partial x^p}{\partial X^s} \frac{\partial X^s}{\partial x^t} = \delta_t^p \quad . \tag{A6}$$

As expected and as it can easily be checked, the expression (A5) is the same as if one starts with the second differential, expressed in terms of the initial coordinates $x^1, x^2, ..., x^n$. This precludes the proof that

$$d^2X^k(X^1, X^2, ..., X^n) = d^2X^i(x^1, x^2, ..., x^n) = \sum_{r=1}^{n} \frac{\partial(dX^k)}{\partial x^r} dx^r \quad . \tag{A7}$$

A2: A PROOF OF THE AFFINE TRANSFORMATION LAW FOR THE CONNECTION $\widetilde{\Gamma}_{ij}^{k}$

Next we proceed with the proof that the defined in (2.11) (and in the preceeding paper [10]) connection

$$\widetilde{\Gamma}_{ij}^{k}=\frac{1}{2}dX^{k}dX^{l}(g_{jl,i}+g_{il,j}-g_{ij,l})=dX^{k}dX^{r}g_{sr}(X)\Gamma_{ij}^{s}(X) \tag{A8}$$

has the transformation property of an affine connection under the coordinate transformations $X^{i}=X^{i}(x^{1},x^{2},...,x^{n})$. The connections $\widetilde{\Gamma}_{ij}^{k}$ and Γ_{ij}^{k} at the spacetime point $\mathbf{X}=(X^{1},X^{2},...,X^{n})$ will be denoted as $\widetilde{\Gamma}_{ij}^{k^{'}}(\mathbf{X})$ and $\Gamma_{ij}^{k^{'}}(\mathbf{X})$, while the same connections at the initial coordinate points $\mathbf{x}=(x^{1},x^{2},...,x^{n})$ will be denoted simply as $\widetilde{\Gamma}_{ij}^{k}(\mathbf{x})$ and $\Gamma_{ij}^{k}(\mathbf{x})$ (the same for the notation $g_{sr}^{'}(\mathbf{X})$).

From the defining equation (A8) for $\widetilde{\Gamma}_{ij}^{k}$, the tensor transformation property for $g_{ij}^{'}(\mathbf{X})$

$$g_{ij}^{'}(\mathbf{X})=\frac{\partial x^{k}}{\partial X^{i}}\frac{\partial x^{l}}{\partial X^{j}}g_{kl}(\mathbf{x}) \quad , \tag{A9}$$

the affine transformation law for the "usual" connection Γ_{ij}^{k}

$$\Gamma_{ij}^{k^{'}}(\mathbf{X})=\Gamma_{np}^{m}(\mathbf{x})\frac{\partial X^{k}}{\partial x^{m}}\frac{\partial x^{n}}{\partial X^{i}}\frac{\partial x^{p}}{\partial X^{j}}+\frac{\partial^{2}x^{m}}{\partial X^{i}\partial X^{j}}\frac{\partial X^{k}}{\partial x^{m}} \tag{A10}$$

and from the expressions for the differentials dX^{k} and dX^{r} we may write down

$$\widetilde{\Gamma}_{ij}^{k^{'}}(\mathbf{X})==dX^{k}(\mathbf{X})dX^{r}(\mathbf{X})g_{sr}^{'}(\mathbf{X})\Gamma_{ij}^{s^{'}}(\mathbf{X})= \tag{A11}$$

$$=\Gamma_{np}^{m}(\mathbf{x})\frac{\partial X^{k}}{\partial x^{\alpha}}\frac{\partial x^{n}}{\partial X^{i}}\frac{\partial x^{p}}{\partial X^{j}}g_{m\beta}(\mathbf{x})dx^{\alpha}dx^{\beta}+\frac{\partial^{2}x^{m}}{\partial X^{i}\partial X^{j}}\frac{\partial X^{k}}{\partial x^{\alpha}}g_{m\beta}(\mathbf{x})dx^{\alpha}dx^{\beta} \quad . \tag{A12}$$

Again, the existence of the inverse transformation and of relation (A6) has been taken into account.

On the other hand, if $\widetilde{\Gamma}_{ij}^{k}(\mathbf{X})$ is an affine connection, then it should satisfy the affine connection transformation law (A10)

$$\widetilde{\Gamma}_{ij}^{k^{'}}(\mathbf{X})=\widetilde{\Gamma}_{np}^{m}(\mathbf{x})\frac{\partial X^{k}}{\partial x^{m}}\frac{\partial x^{n}}{\partial X^{i}}\frac{\partial x^{p}}{\partial X^{j}}+\frac{\partial^{2}x^{m}}{\partial X^{i}\partial X^{j}}\frac{\partial X^{k}}{\partial x^{m}} \quad . \tag{A13}$$

Making use of the defining equation (A8) (but in terms of the initial coordinates $x^{1},x^{2},....,x^{n}$), the above expression can be written also as

$$\widetilde{\Gamma}_{ij}^{k^{'}}(\mathbf{X})=\Gamma_{np}^{m}(\mathbf{x})\frac{\partial X^{k}}{\partial x^{\alpha}}\frac{\partial x^{n}}{\partial X^{i}}\frac{\partial x^{p}}{\partial X^{j}}g_{m\beta}(\mathbf{x})dx^{\alpha}dx^{\beta}+\frac{\partial^{2}x^{\alpha}}{\partial X^{i}\partial X^{j}}\frac{\partial X^{k}}{\partial x^{\alpha}} \quad . \tag{A14}$$

Clearly, if $\widetilde{\Gamma}^{k'}_{ij}(\mathbf{X})$ is an affine connection, from the R. H. S. of (A12) and (A14) it would follow that the following relation has to be satisfied

$$dx^{\alpha}dx^{\beta}g_{m\beta}(\mathbf{x})\frac{\partial^2 x^m}{\partial X^i \partial X^j}\frac{\partial X^k}{\partial x^{\alpha}} - \frac{\partial^2 x^{\alpha}}{\partial X^i \partial X^j}\frac{\partial X^k}{\partial x^{\alpha}} = 0 \quad . \tag{A15}$$

Let us first prove that the second term is equal to zero. We have

$$\frac{\partial X^k}{\partial x^{\alpha}}\frac{\partial^2 x^{\alpha}}{\partial X^i \partial X^j} = \frac{\partial}{\partial X^i}\left[\frac{\partial X^k}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial X^j}\right] - \frac{\partial x^{\alpha}}{\partial X^j}\frac{\partial}{\partial x^{\alpha}}\left[\frac{\partial X^k}{\partial X^i}\right] =$$

$$= \frac{\partial}{\partial X^i}\delta^k_j - \frac{\partial x^{\alpha}}{\partial X^j}\frac{\partial}{\partial x^{\alpha}}\delta^k_i = 0 \quad . \tag{A16}$$

Making use of the tensor transformation property (A9), the first term in (A15) can be transformed as follows

$$dx^{\alpha}dx^{\beta}g_{m\beta}(\mathbf{x})\frac{\partial^2 x^m}{\partial X^i \partial X^j}\frac{\partial X^k}{\partial x^{\alpha}} = \tag{A17}$$

$$= \frac{\partial x^{\alpha}}{\partial X^r}\frac{\partial x^{\beta}}{\partial X^s}dX^r dX^s \frac{\partial X^p}{\partial x^m}\frac{\partial X^q}{\partial x^{\beta}}\frac{\partial X^k}{\partial x^{\alpha}}\frac{\partial^2 x^m}{\partial X^i \partial X^j} = \tag{A18}$$

$$= dX^r dX^s \delta^k_r \delta^q_s \frac{\partial X^p}{\partial x^m}\frac{\partial^2 x^m}{\partial X^i \partial X^j} \quad . \tag{A19}$$

But the above expression has already been proved to be equal to zero. Therefore, equation (A15) is satisfied and consequently, $\widetilde{\Gamma}^k_{ij}$ has an affine connection transformation property.

A3: THE CONNECTION $\widetilde{\Gamma}^k_{ij}$ AS AN EQUIAFFINE CONNECTION

We have to prove that the connection $\widetilde{\Gamma}^k_{ij}$ for $j = k$ can be represented in the form of a gradient of a scalar quantity, i. e.

$$\widetilde{\Gamma}^k_{ij} = \partial_i lne \quad . \tag{A20.}$$

From the defining formulae (A8), it follows

$$\widetilde{\Gamma}^k_{ij} = \frac{1}{2}\left[dX^k dX^s g_{ks}\right]_{,i} - \frac{1}{2}g_{is}\left[dX^k dX^s\right]_{,i} +$$

$$+\frac{1}{2}\left[dX^k dX^s g_{is}\right]_{,k} - \frac{1}{2}g_{is}\left[dX^k dX^s\right]_{,k} -$$

$$-\frac{1}{2}\left[dX^k dX^s g_{ik}\right]_{,s} - \frac{1}{2}g_{ik}\left[dX^k dX^s\right]_{,s} \quad . \tag{A21.}$$

The last four terms cancel, so therefore from the first two terms it is evident that in the approximation $(dX^i)_{,k} = 0$ the connection $\widetilde{\Gamma}_{ij}^{k}$ is indeed an equiaffine one, since one can set up

$$lne \equiv \frac{1}{2} dX^k dX^s g_{ks} \tag{A22}$$

and so e will be fully determined and the defining equality (A20) then will be fulfilled.

The more complicated and interesting task is to prove that even in the case $(dX^i)_{,k} \neq 0$, the connection $\widetilde{\Gamma}_{ij}^{k}$ will again be an equiaffine one. For the purpose, note that

$$\widetilde{\Gamma}_{ik}^{k} = \frac{1}{2} dX^s dX^k g_{ks,i} = \frac{1}{2} dX^k dX^s g_{r(s}\Gamma_{k)i}^{r} = W_i \tag{A23.}$$

and consequently $\widetilde{\Gamma}_{ik}^{k}$ will be an equiaffine connection if the scalar quantity e can be determined as a solution of the differential equation

$$\partial_i lne = W_i \tag{A24.}$$

as

$$e = g(X_1, X_2, .., X_{i-1}, X_{i+1}, .., X_n) e^{\int W_i(X_1, ..., X_n) dX^i} \quad . \tag{A25.}$$

Note that the function g depends on all variables $X_1, X_2, .., X_{i-1}, X_{i+1}, .., X_n$ with the exception of X_i, while the function W_i depends on all the variables, including also X_i.

Unfortunately, the proof at this stage will be incomplete, since e will depend on the choice of the variable X_i, which should not happen with a scalar quantity.Consequently, it should be proved that the function $g(X_1, X_2, .., X_{i-1}, X_{i+1}, .., X_n)$ can be determined in a proper way so (that for every choice of W_i the expression (A25) for e would be a scalar quantity. Until we have not proved it, we shall denote the L.H. S. of (A25) with $e^{(i)}$.

Let us differentiate both sides of (A25) for $e \equiv e^{(i)}$ and $e \equiv e^{(j)}$ by X^j and X^i respectively $(i \neq j)$. We shall write down only the first equation, since the second one is obtained from the first after a change of the indices $i \Longleftrightarrow j$.

$$\frac{\partial e^{(i)}}{\partial X^j} = \frac{\partial \ln g(X_1, X_2, ., X_{i-1}, X_{i+1}, ., X_n)}{\partial X^j} e^{(i)} +$$

$$+ g(X_1, X_2, .., X_{i-1}, X_{i+1}, .., X_n) e^{\int \frac{\partial W_i(X_1, ..., X_n)}{\partial X^j} dX^i} \quad . \tag{A26}$$

Now differentiate again the derived equation (A26) for $\frac{\partial e^{(i)}}{\partial X^j}$ by X^i and the other equation for $\frac{\partial e^{(j)}}{\partial X^i}$ by X^j. Taking into account also that $\frac{\partial e^{(i)}}{\partial X^i} = e^{(i)} W_i$,the result for the first equation will be

$$\frac{\partial^2 e^{(i)}}{\partial X^j \partial X^i} = \frac{\partial \ln g(X_1, X_2, ., X_{i-1}, X_{i+1}, ., X_n)}{\partial X^j} e^{(i)} W_i +$$

$$+ g(X_1, X_2, .., X_{i-1}, X_{i+1}, .., X_n) \frac{\partial W_i}{\partial X^j} e^{\int \frac{\partial W_i(X_1, ..., X_n)}{\partial X^j} dX^i} \quad . \tag{A27}$$

In respect to the second term in (A27), again the equality (A26) may be applied and after that a summation along the indices i and j can be defined. Equation (A27) acquires the form

$$\sum_{i,j}\frac{\partial}{\partial X^j}\left(\frac{\partial e^{(i)}}{\partial X^i}\right)=grad\left[\ln g(X_1,X_2,.,X_{i-1},X_{i+1},.,X_n)\right](\mathbf{e}\;.\mathbf{W})+$$

$$+\sum_{i,j}\left[\frac{\partial W_i}{\partial X^j}\frac{\partial e^{(i)}}{\partial X^j}-\frac{\partial\widetilde{W}_{ij}}{\partial X^j}.e^{(i)}\right]+(\mathbf{e}\;.\mathbf{W})\triangle\ln g(X_1,X_2,.,X_{i-1},X_{i+1},.,X_n)\qquad,\qquad(A28)$$

where $(\mathbf{e}\;.\mathbf{W})$ denotes a scalar product and the following notations have been introduced

$$\triangle\ln g(....)\equiv\sum_j\frac{\partial^2}{\partial X^{j2}}\triangle\ln g(.....)\qquad,\qquad(A29.)$$

$$\widetilde{W}_{ij}\equiv W_i\frac{\partial\ln g(X_1,X_2,.,X_{i-1},X_{i+1},.,X_n)}{\partial X^j}\qquad.\qquad(A30)$$

Again, the second equation will be the same as (A28), but with $i\Leftrightarrow j$. Substracting the two equations and taking into account that

$$graddive=\sum_{i,j}\frac{\partial}{\partial X^j}\left(\frac{\partial e^{(i)}}{\partial X^i}\right)=\sum_{i,j}\frac{\partial}{\partial X^i}\left(\frac{\partial e^{(i)}}{\partial X^j}\right)\qquad,\qquad(A31.)$$

one can derive

$$\sum_{i,j}\left[\frac{\partial W_i}{\partial X^j}\frac{\partial e^{(i)}}{\partial X^i}-\frac{\partial W_j}{\partial X^i}\left(\frac{\partial e^{(j)}}{\partial X^i}\right)\right]-$$

$$-\sum_{i,j}\left[\frac{\partial\widetilde{W}_{ij}}{\partial X^j}e^{(i)}-\frac{\partial\widetilde{W}_{ji}}{\partial X^i}e^{(j)}\right]=0\qquad.\qquad(A32.)$$

But it can be written also

$$\sum_{i,j}\left[\frac{\partial W_i}{\partial X^j}\frac{\partial e^{(i)}}{\partial X^i}-\frac{\partial\widetilde{W}_{ij}}{\partial X^j}e^{(i)}\right]=$$

$$=\sum_{i,j}\left[\frac{\partial W_i}{\partial X^j}\frac{\partial e^{(i)}}{\partial X^i}-e^{(i)}\frac{\partial W_i}{\partial X^j}\frac{\partial\ln g}{\partial X^j}-(\mathbf{e}\;.\mathbf{W})\triangle\ln g\right]\qquad.\qquad(A33.)$$

Taking the above expression into account, equation (A32) acquires the form

$$\sum_{i,j}\left[\frac{\partial W_i}{\partial X^j}\frac{\partial e^{(i)}}{\partial X^i}-\frac{\partial W_i}{\partial X^j}\frac{\partial e^{(i)}}{\partial X^i}\right]+$$

$$+\sum_{i,j}\left[e^{(j)}\frac{\partial W_j}{\partial X^i}\frac{\partial\ln g}{\partial X^i}-e^{(i)}\frac{\partial W_i}{\partial X^j}\frac{\partial\ln g}{\partial X^j}\right]=0\qquad.\qquad(A34.)$$

Further we shall assume that each term in the sum is zero, i.e. the equation is fulfilled for each i and j. Substituting expressions (A25) for $e^{(i)}$ and $e^{(j)}$ and (A26) for $\frac{\partial e^{(i)}}{\partial X^j}$ and $\frac{\partial e^{(j)}}{\partial X^i}$, equation (A34) can be rewritten in the following simple form:

$$\frac{\partial W_i}{\partial X^j} g(X_1,..,X_{i-1},X_{i+1},..,X_n) e^{\int \frac{\partial W_i}{\partial X^j} dX^i} =$$

$$= \frac{\partial W_j}{\partial X^i} g(X_1,..,X_{j-1},X_{j+1},..,X_n) e^{\int \frac{\partial W_j}{\partial X^i} dX^j} \quad . \tag{A35.}$$

Differentiating this expression by X^i and making use again of (A25) and (A26), the following simple differential equation can be obtained:

$$W_{j,i} \frac{\partial \ln g(X_1,.,X_{j-1},X_{j+1},..,X_n)}{\partial X^i} + (W_{j,ii} - W_{i,j}W_{j,i} -$$

$$- \frac{W_{i,ji}W_{j,i}}{W_{i,j}}) + W_{j,i} e^{\int \left[\frac{\partial^2 W_j}{\partial X^{i2}} - \frac{\partial W_j}{\partial X^i} \right] dX^j} = 0 \quad . \tag{A36.}$$

The first case, when this equation will be satisfied will be

$$W_{j,i} = \frac{\partial W_j}{\partial X^i} = 0 \quad \Longrightarrow W_j = f(X_1,..,X_{i-1},X_{i+1},..,X_n) \quad . \tag{A37.}$$

Since this will be fulfilled for **every** i, then W_j should be a constant, which of course is a very rare and special case.

The second, more realistic case is when the function g is a solution of the differential equation (A36) for every i and j ($i \neq j$):

$$g(X_1,.,X_{j-1},X_{j+1},..,X_n) = F(X_1,.,X_{i-1},X_{i+1},..,X_n) e^{\int \widetilde{Q}(X_1,...,X_n) dX^i} \quad , \tag{A38.}$$

where

$$\widetilde{Q}(X_1,...,X_n) \equiv \left(W_{i,j} + \frac{W_{i,ji}}{W_{i,j}} - \frac{W_{j,ii}}{W_{j,i}} \right) - e^{\int \left(\frac{\partial^2 W_j}{\partial X^{i2}} - \frac{\partial W_j}{\partial X^i} \right) dX^j} \quad . \tag{A39.}$$

Since the function $g(X_1,.,X_{j-1},X_{j+1},..,X_n)$ on the L. H. S. of (A38) does not depend on the variable X_j, then for each j the unknown function $F(X_1,.,X_{i-1},X_{i+1},..,X_n)$ can be obtained after differentiating both sides of (A38) by X^j. Thus the function F is a solution of the following differential equation

$$0 = \frac{\partial F(X_1,.,X_{i-1},X_{i+1},..,X_n)}{\partial X^j} e^{\int \widetilde{Q} dX^i} + F(X_1,.,X_{i-1},X_{i+1},..,X_n) e^{\int \frac{\partial \widetilde{Q}}{\partial X^j} dX^i} \quad . \tag{A40.}$$

This precludes the proof that the function g in (A25) can be determined in such a way that $e^{(i)}$ would be indeed a scalar quantity and therefore $e \equiv e^{(i)}$. Throughout the whole proof, we assumed that W_i, determined by (A23), is a vector. This of course should be proved in the same way, in which it was proved that the connection $\widetilde{\Gamma}_{ij}^k$ has affine transformation properties.

12 APPENDIX B: SOME COEFFICIENT FUNCTIONS IN THE FINAL SOLUTIONS FOR dX^1, dX^2, dX^3 IN SECTION III

The functions h_1, h_2, h_3 (depending on the Weierstrass function $\rho(z)$) and the functions l_1, l_2, l_2 (not depending on $\rho(z)$) in the expression (3.21) for the solution dX^2 of the cubic algebraic equation are

$$h_1 \equiv 2p\left(\frac{b_2}{c_2} - 2\frac{b_2}{d_2}L_1^{(2)}\rho(z)\right)(2\Gamma^r_{12}g_{1r} + \Gamma^r_{11}g_{2r}) \quad , \tag{B1.}$$

$$h_2 \equiv \left(\frac{b_2}{c_2} - 2\frac{b_2}{d_2}L_1^{(2)}\rho(z)\right)\left(K_{12}^{(1)} + K_{21}^{(1)}\right) -$$

$$-2pL_1^{(2)}\rho(z)\,(2\Gamma^r_{12}g_{2r} + \Gamma^r_{22}g_{1r}) \quad , \tag{B2.}$$

$$h_3 \equiv \left(\frac{b_2}{c_2} - 2\frac{b_2}{d_2}L_1^{(2)}\rho(z)\right)K_2^{(2)} - L_1^{(2)}\rho(z)K_{22}^{(1)} \quad , \tag{B3.}$$

$$l_1 \equiv 2p\left(\frac{d_2}{c_2} - 2\frac{b_2}{d_2}L_1^{(2)}\right)(2\Gamma^r_{12}g_{1r} + \Gamma^r_{11}g_{2r}) \quad , \tag{B4.}$$

$$l_2 \equiv \left(\frac{d_2}{c_2} - 2\frac{b_2}{d_2}L_1^{(2)}\right)\left(K_{12}^{(1)} + K_{21}^{(1)}\right) -$$

$$-2pL_1^{(2)}\,(2\Gamma^r_{12}g_{2r} + \Gamma^r_{22}g_{1r}) \quad , \tag{B5.}$$

$$l_3 \equiv \left(\frac{d_2}{c_2} - 2\frac{b_2}{d_2}L_1^{(2)}\right)K_2^{(2)} - L_1^{(2)}K_{22}^{(1)} \quad . \tag{B6.}$$

The functions $F_1, F_2, F_3, f_1, f_2, f_3, f_4, \widetilde{g}_1, \widetilde{g}_2, \widetilde{g}_3$ in the solution for dX^1 are the following

$$F_1 \equiv 2p\,(2\Gamma^r_{12}g_{2r} + \Gamma^r_{22}g_{1r}) \quad ; \quad F_2 \equiv \left(1 + 2\frac{d_2}{c_2}\right)\left(K_{12}^{(1)} + K_{21}^{(1)}\right) \quad , \tag{B7.}$$

$$F_3 \equiv 2p\left(1 + 2\frac{d_2}{c_2}\right)(2\Gamma^r_{12}g_{1r} + \Gamma^r_{11}g_{2r}) \quad , \tag{B8.}$$

$$f_1 \equiv -2\frac{b_1}{d_1}L_1^{(1)}F_1 \quad ; \quad f_3 \equiv \frac{b_1}{c_1}F_2 - L_1^{(1)} \quad , \tag{B9.}$$

$$f_2 \equiv \frac{b_1}{c_1}F_1 - L_1^{(1)}F_3 - 2\frac{b_1}{d_1}L_1^{(1)}F_2 \quad , \tag{B10.}$$

$$\widetilde{g}_1 \equiv \left(\frac{d_1}{c_1} - 2\frac{b_1}{d_1}\right)F_1 \quad , \tag{B11.}$$

$$\widetilde{g}_2 \equiv \left(\frac{d_1}{c_1} - 2\frac{b_1}{d_1}\right)F_2 - L_1^{(1)}F_3 \quad , \tag{B12.}$$

$$\widetilde{g}_3 \equiv \left(\frac{d_1}{c_1} - 2\frac{b_1}{d_1}\right)K_1^{(2)} - L_1^{(1)}K_{11}^{(1)} \quad . \tag{B13.}$$

13 APPENDIX C: BLOCK STRUCTURE METHOD FOR SOLVING THE SYSTEM OF EQUATIONS $g_{ij}g^{jk} = \delta_i^k$ IN THE GENERAL N - DIMENSIONAL CASE.

C1. BLOCK STRUCTURE METHOD FOR THE PARTIAL $n = 3$ CASE

The purpose of the present section will be to develop a method for solving the system of equations $g_{ij}g^{jk} = \delta_i^k$ for the case $n = 3$. Since in principle the system for $n = 3$ can be solved in an elementary manner, the aim will be not to find another more convenient method, but rather than that find a method, which can further be generalized to higher dimensions.

Let us first write down the system of six equations for different indices i and k :

$$g_{11}g^{12} + g_{12}g^{22} + g_{13}g^{32} = 0 \quad , \tag{C1.}$$

$$g_{21}g^{11} + g_{22}g^{21} + g_{23}g^{31} = 0 \quad , \tag{C2.}$$

$$g_{11}g^{13} + g_{12}g^{23} + g_{13}g^{33} = 0 \quad , \tag{C3.}$$

$$g_{31}g^{11} + g_{32}g^{21} + g_{33}g^{31} = 0 \quad , \tag{C4.}$$

$$g_{21}g^{13} + g_{22}g^{23} + g_{23}g^{33} = 0 \quad , \tag{C5.}$$

$$g_{31}g^{12} + g_{32}g^{22} + g_{33}g^{32} = 0 \quad . \tag{C6.}$$

In matrix notations and for the unknown variables $g_{11}, g_{22}, g_{33}, g_{12}, g_{23}$ and g_{13} the system can be written as

$$AX = 0 \quad , \tag{C7.}$$

where A is the 6×6 dimensional matrix

$$A \equiv \begin{pmatrix} g^{12} & 0 & 0 & g^{22} & 0 & g^{32} \\ 0 & g^{21} & 0 & g^{11} & g^{31} & 0 \\ g^{13} & 0 & 0 & g^{23} & 0 & g^{33} \\ 0 & 0 & g^{31} & 0 & g^{21} & g^{11} \\ 0 & g^{23} & 0 & g^{13} & g^{33} & 0 \\ 0 & 0 & g^{32} & 0 & g^{22} & g^{12} \end{pmatrix} \tag{C8.}$$

and the transponed vector X^T is

$$X^T \equiv (g_{11}, g_{22}, g_{33}, g_{12}, g_{23}, g_{13}) \quad . \tag{C9.}$$

By a direct calculation it can easily be checked that $detA = 0$ and therefore the homogeneous sysytem of equations (C7) has arbitrary solutions.

Now let us write down the rest of the system of equations for the case $i = k$, which in matrix notations reads as

$$BX = \overline{1}^T \quad , \qquad \text{(C10.)}$$

where $\overline{1}^T = (1,1,1)$ and B is the 6×3 matrix

$$B \equiv \begin{pmatrix} g^{11} & 0 & 0 & g^{12} & 0 & g^{13} \\ 0 & g^{22} & 0 & g^{12} & g^{23} & 0 \\ 0 & 0 & g^{33} & 0 & g^{23} & g^{13} \end{pmatrix} \quad . \qquad \text{(C11.)}$$

Since the system (C10) can be written either as

$$B_1 X = B_2 \qquad or \qquad B_2 X = B_1 \quad , \qquad \text{(C12.)}$$

$$B_1 \equiv \begin{pmatrix} g^{11} & 0 & 0 \\ 0 & g^{22} & 0 \\ 0 & 0 & g^{33} \end{pmatrix} \quad ; \quad B_2 \equiv \begin{pmatrix} g^{12} & 0 & g^{13} \\ g^{12} & g^{23} & 0 \\ 0 & g^{23} & g^{13} \end{pmatrix} \qquad \text{(C13.)}$$

and $detB_1 \neq 0$; $detB_2 \neq 0$, it is clear that one can choose freely either the three components (g_{11}, g_{22}, g_{33}) or (g_{12}, g_{23}, g_{13}) and after that **fix the other three components** by finding the unique solutions of the system (C10). Also, it is important to note that **the above conclusions do not change if the contravariant metric tensor is chosen in the form of the factorized product** $g^{ij} = dX^i dX^j$**.**

Now it is worth noting that the entire system of equations $g_{ij}g^{jk} = \delta_i^k$ for $n = 3$ can be written as

$$\widetilde{A}X = E \quad , \qquad \text{(C13.)}$$

where E^T is the transponed $9-$ vector:

$$E^T = (1,0,0,0,1,1,0,0,1) \quad . \qquad \text{(C14.)}$$

The 6×9 matrix $\widetilde{A}$ has the following interesting block structure:

$$\widetilde{A} \equiv \begin{pmatrix} P_1 & Q_1 \\ P_2 & Q_2 \\ P_3 & Q_3 \end{pmatrix} \quad , \qquad \text{(C15.)}$$

where $(s = 1,2,3)$ the matrices P_s and Q_s are the following:

$$P_s \equiv \begin{pmatrix} g^{s1} & g^{s2} & g^{s3} \\ 0 & g^{s1} & 0 \\ 0 & 0 & g^{s1} \end{pmatrix} \quad ; \quad Q_s \equiv \begin{pmatrix} 0 & 0 & 0 \\ g^{2s} & g^{3s} & 0 \\ 0 & g^{2s} & g^{3s} \end{pmatrix} \quad . \qquad \text{(C16.)}$$

In order to find the solution $X = \widetilde{A}^{-1}E$ of the system (C13), one has to find the inverse matrix $\widetilde{A}^{-1}$. For the case of **quadratic** matrices with the block structure

$$M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \qquad \text{(C17.)}$$

where A, B, C and D are $n \times n, q \times n, n \times q$ and $q \times q$ matrices correspondingly, the so called **Frobenius formulae** [20] for finding the inverse matrix M^{-1} is valid

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BH^{-1}CA^{-1} & -A^{-1}BH^{-1} \\ -H^{-1}CA^{-1} & H^{-1} \end{pmatrix} \quad , \tag{C18.}$$

where H is the matrix:

$$H \equiv D - CA^{-1}B \quad . \tag{C19.}$$

In the present case, the **Frobenius formulae cannot be applied to the block matrix (C15), since it is not a quadratic one.**However, if X is a solution of the system (C13), then it is a solution also of the equation $\widetilde{A}^T\widetilde{A}X = E\widetilde{A}^T$, where the 6×9 matrix $\widetilde{A}$, multiplied to the left with its transponed one, gives already the quadratic 9×9 matrix $\widetilde{A}^T\widetilde{A}$. Further it shall be demonstrated how the Frobenius inversion formulae can be applied twice in respect to $\widetilde{A}^T\widetilde{A}$.

The matrix $\widetilde{A}^T\widetilde{A}$ can be calculated to be the following block matrix:

$$\widetilde{A}^T\widetilde{A} = \begin{pmatrix} \sum\limits_{i=1}^{3} P_i^T P_i & \sum\limits_{j=1}^{3} P_j^T Q_j \\ \sum\limits_{k=1}^{3} Q_k^T P_k & \sum\limits_{l=1}^{3} Q_l^T Q_l \end{pmatrix} \tag{C20.}$$

and P_i, Q_j are the corresponding matrices (C15) and their transponed ones. The block matrices in (C20) are found to be the following 3×3 matrices, which shall further be identified with the corresponding block - matrices A, B, C and D in (C17):

$$A \equiv \sum_{i=1}^{3} P_i^T P_i = \begin{pmatrix} \sum\limits_{i=1}^{3}(g^{i1})^2 & \sum\limits_{i=1}^{3} g^{i1}g^{i2} & \sum\limits_{i=1}^{3} g^{i1}g^{i3} \\ \sum\limits_{i=1}^{3} g^{i1}g^{i2} & \sum\limits_{i=1}^{3}[(g^{i1})^2 + (g^{i2})^2] & \sum\limits_{i=1}^{3} g^{i2}g^{i3} \\ \sum\limits_{i=1}^{3} g^{i1}g^{i3} & \sum\limits_{i=1}^{3} g^{i2}g^{i3} & \sum\limits_{i=1}^{3}[(g^{i1})^2 + (g^{i3})^2] \end{pmatrix} \quad , \tag{C21.}$$

$$B \equiv \sum_{j=1}^{3} P_j^T Q_j = \sum_{j=1}^{3} g^{j1} \begin{pmatrix} 0 & 0 & 0 \\ g^{2j} & g^{3j} & 0 \\ 0 & g^{j2} & g^{j3} \end{pmatrix} \quad , \tag{C22.}$$

$$C \equiv \sum_{k=1}^{3} Q_k^T P_k = \sum_{k=1}^{3} g^{k1} \begin{pmatrix} 0 & g^{k2} & 0 \\ 0 & g^{k3} & g^{k2} \\ 0 & 0 & g^{k3} \end{pmatrix} \quad , \tag{C23.}$$

$$D \equiv \sum_{l=1}^{3} Q_l^T Q_l = \begin{pmatrix} \sum\limits_{l=1}^{3}(g^{2l})^2 & \sum\limits_{l=1}^{3} g^{2l}g^{3l} & 0 \\ \sum\limits_{l=1}^{3} g^{2l}g^{3l} & \sum\limits_{l=1}^{3}\left[(g^{21})^2 + (g^{3l})^2\right] & \sum\limits_{l=1}^{3} g^{2l}g^{3l} \\ 0 & \sum\limits_{l=1}^{3} g^{2l}g^{3l} & \sum\limits_{i=1}^{3}(g^{3l})^2 \end{pmatrix} \quad . \tag{C24.}$$

Note that the diagonal block - matrices A and D have non - zero determinants (even if $g^{ij} = dX^i dX^j$), while the non - diagonal block - matrices B and D have zero - determinants. However, in order to apply the Frobenius formulae for inverting the matrix (C20) it is sufficient to have as invertible only the matrix A (and of course the matrix H).

C2. MODIFICATION OF THE BLOCK STRUCTURE OF THE MATRIX A

The above presented method has nevertheless the following shortcomings:

1. It deals with an rectangular $p \times q$ matrix A for a system of equations with $p = \begin{pmatrix} n \\ 2 \end{pmatrix} + n$ variables and $q = n^2$ equations. At the same time it would have been much better to deal with a quadratic matrix at the beginning.

2. The block - matrix A contains two types of matrices P_s and Q_s,while it would be more convenient to have just one type of an elementary "constituent" $E_k^{(i)}$ with a definite structure, where the indice i denotes the corresponding column in the block matrice A (i.e. the number of the column, containing block - matrices) and the indice k - the corresponding row of block - matrices.

3. The matrix Q_s, given by the formulae (C15) has a zero determinant.

To avoid these shortcomings, let us define an extended $9-$ dimensional vector Y, whose transponed one is

$$Y^T \equiv (g_{11}, g_{12}, g_{13}, g_{21}, g_{22}, g_{23}, g_{31}, g_{32}, g_{33}) \quad , \tag{C26.}$$

where the elements g_{21}, g_{32} and g_{31} formally shall be considered unknown, although they are equal to their symmetric counterparts g_{12}, g_{23} and g_{13}. Then the system of equations can be written as

$$MY = \overline{1} \quad , \tag{C27.}$$

where $\overline{1}^T$ is the transponed 9 -dimensional vector

$$\overline{1}^T \equiv (1, 0, 0, 0, 1, 0, 0, 0, 1) \tag{C28.}$$

and the 9×9 matrix M has the following block structure

$$M \equiv \begin{pmatrix} E_1^{(1)} & E_1^{(2)} & E_1^{(3)} \\ E_2^{(1)} & E_2^{(2)} & E_2^{(3)} \\ E_3^{(1)} & E_3^{(2)} & E_3^{(3)} \end{pmatrix} \quad . \tag{C29.}$$

The elementary 3×3 block matrices $E_k^{(1)}$, $E_k^{(2)}$ and $E_k^{(3)}$ ($k = 1, 2, 3$ denotes the number of the row) in each column are the following

$$E_k^{(1)} \equiv \frac{1}{2} \begin{pmatrix} 2g^{k1} & g^{k2} & g^{k3} \\ 0 & g^{k1} & 0 \\ 0 & 0 & g^{k1} \end{pmatrix} \quad , \tag{C30.}$$

$$E_k^{(2)} \equiv \frac{1}{2} \begin{pmatrix} g^{2k} & 0 & 0 \\ g^{1k} & 2g^{2k} & g^{3k} \\ 0 & 0 & g^{2k} \end{pmatrix} \quad , \tag{C31.}$$

$$E_k^{(3)} \equiv \frac{1}{2} \begin{pmatrix} g^{3k} & 0 & 0 \\ 0 & g^{3k} & 0 \\ g^{1k} & g^{2k} & g^{3k} \end{pmatrix} \quad . \tag{C32.}$$

From (C30 - C32) it is seen that depending on the indice k of the elementary matrices row and of the indice (s) for the elementary matrices column, the matrix $E_k^{(s)}$ for the $n-$ dimensional case can be written as

$$E_k^{(s)} \equiv \frac{1}{(n-1)} \begin{pmatrix} g^{ks} & 0... & .... & ... & 0 & 0 \\ 0 & g^{ks} & 0.. & ... & 0 & 0 \\ .... & .... & ... & .... & .... & ... \\ g^{k1} & g^{k2}. & .. & (n-1)g^{ks}.. & ... & g^{kn} \\ .... & ... & ... & .... & .... & .... \\ 0 & 0 & 0 & .... & 0 & g^{ks} \end{pmatrix} \quad . \tag{C33.}$$

In other words, on the main diagonal of the matrix $E_k^{(s)}$ the elements g^{ks} are situated, on the $s-$th row - the elements $\left(\; g^{k1}, \;\; g^{k2}, . \;\; .. \;\; (n-1)g^{ks} .. \;\; ... \;\; , g^{kn} \; \right)$ with an element $(n-1)g^{ks}$ on the $s-$th row and $s-$th column.This block structure (with some slight modifications) shall be obtained also for the $n-$ dimensional case, and thus the $3-$ dimensional case really helps to make the corresponding generalization for the $n-$ dimensional case.

Another advantage of the block - matrice representation (C29) is that it gives a possibility to apply **twice** the Frobenius formulae. Correspondingly, in the $n-$ dimensional case the Frobenius formulae will be applied $(n-1)$ times.

For the representation (C29) this can be performed in the following two steps:

Step 1. Apply the Frobenius formulae to the sub - matrix $\widetilde{A}$ of the matrix (C29), where $\widetilde{A}$ is the matrix of the first two columns and first two rows

$$\widetilde{A} \equiv \begin{pmatrix} E_1^{(1)} & E_1^{(2)} \\ E_2^{(1)} & E_2^{(2)} \end{pmatrix} \tag{C34.}$$

The inverse matrix $\widetilde{A}^{-1}$ will be given by the formulae (C18) with the following identifications

$$A \equiv E_1^{(1)}; \qquad B \equiv E_1^{(2)} \quad ; \quad C \equiv E_2^{(1)} \quad ; \quad D \equiv E^{(2)} \quad . \tag{C35.}$$

Step 2. Denote by $\widetilde{B}, \widetilde{C}$ and $\widetilde{D}$ the following $3 \times 6, 6 \times 3$ and 3×3 matrices:

$$\widetilde{B} \equiv \begin{pmatrix} E_1^{(3)} \\ E_2^{(3)} \end{pmatrix}; \qquad \widetilde{C} \equiv \begin{pmatrix} E_3^{(1)} & E_3^{(2)} \end{pmatrix} \quad ; \qquad \widetilde{D} \equiv E_3^{(3)} \quad . \tag{C36.}$$

Therefore the matrix M is written as

$$M \equiv \begin{pmatrix} \widetilde{A} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix} \tag{C37.}$$

and the Frobenius formulae is applied again to the block - matrix M, resulting in the inverse matrix

$$M^{-1} = \begin{pmatrix} \widetilde{A}^{-1} + \widetilde{A}^{-1}\widetilde{B}\widetilde{H}^{-1}\widetilde{C}\widetilde{A}^{-1} & -\widetilde{A}^{-1}\widetilde{B}\widetilde{H}^{-1} \\ -\widetilde{H}^{-1}\widetilde{C}\widetilde{A}^{-1} & \widetilde{H}^{-1} \end{pmatrix} \quad , \tag{C38.}$$

where $\widetilde{H}$ is the matrix:

$$\widetilde{H} \equiv \widetilde{D} - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} \tag{C39.}$$

and $\widetilde{A}^{-1}$ is the calculated inverse matrix from step 1.

Now let us find the matrix $\widetilde{A}^{-1}\widetilde{B}$ in terms of the matrices $E_k^{(s)}$:

$$\widetilde{A}^{-1}\widetilde{B} = \begin{pmatrix} E_1^{(1)-1} + E_1^{(1)-1}E_1^{(2)}H_1^{-1}E_2^{(1)}E_1^{(1)-1} & -E_1^{(1)}E_1^{(2)}H_1^{-1} \\ -H_1^{-1}E_2^{(1)}E_1^{(1)-1} & H_1^{-1} \end{pmatrix} \begin{pmatrix} E_1^{(3)} \\ E_2^{(3)} \end{pmatrix} \quad , \tag{C40.}$$

where H_1 is given by

$$H_1 \equiv E_2^{(2)} - E_2^{(1)}E_1^{(1)-1}E_1^{(2)} \tag{C41.}$$

and is obviously different from $\widetilde{H}$ in (C39). Finally $\widetilde{A}^{-1}\widetilde{B}$ is calculated to be the following matrix - column:

$$\widetilde{A}^{-1}\widetilde{B} = \begin{pmatrix} E_2^{(1)-1}E_2^{(2)}E_1^{(2)-1}E_1^{(3)} - E_1^{(1)-1}E_1^{(2)}X^{-1}E_1^{(1)}E_2^{(1)-1}E_2^{(3)} \\ -X^{-1}E_1^{(3)} + X^{-1}E_1^{(1)}E_2^{(1)-1}E_2^{(3)} \end{pmatrix} \tag{C42.}$$

with

$$X \equiv E_1^{(1)}E_2^{(1)-1}E_2^{(2)} - E_1^{(2)} \qquad . \tag{C43.}$$

Note the obvious advantage of the method, in which all the sub - matrices are invertible in comparison with the method in the previous sub-section (C1), when only the diagonal submatrices had been invertible. For example, in the present case it had been used that block element $E_1^{(1)}E_1^{(2)}H_1^{-1}$ in (C40) can be represented as

$$-E_1^{(1)}E_1^{(2)}H_1^{-1} = -(H_1E_1^{(2)-1}E_1^{(1)})^{-1} \quad , \tag{C44.}$$

which obviously cannot be applied in the previous case.

Further the expression for $\widetilde{H}$ can be found, which participates in the formulae for M^{-1} :

$$\widetilde{H} \equiv \widetilde{D} - \widetilde{C}\widetilde{A}^{-1}\widetilde{B} = E_3^{(3)} - E_3^{(1)}\{E_2^{(1)-1}E_2^{(2)}E_1^{(2)-1}E_1^{(3)} -$$

$$-E_1^{(1)-1}E_1^{(2)}X^{-1}E_1^{(1)}E_2^{(1)-1}E_2^{(3)}\} - E_3^{(2)}X^{-1}\{-E_1^{(3)} + E_1^{(1)}E_2^{(1)-1}E_2^{(3)}\} \quad . \tag{C45.}$$

In order to calculate M^{-1} according to (C38), one needs to find also the expression for $\widetilde{C}\widetilde{A}^{-1}$:

$$\widetilde{C}\widetilde{A}^{-1} = \begin{pmatrix} E_3^{(1)}E_1^{(1)-1} - E_3^{(1)}E_1^{(1)-1}E_1^{(2)}X^{-1} - E_3^{(2)}X^{-1} \\ [E_3^{(2)} - E_3^{(1)}E_1^{(1)-1}E_1^{(2)}](X^{-1}E_1^{(1)}E_2^{(1)-1} \end{pmatrix} \quad . \tag{C46.}$$

The entire calculation for M^{-1} is rather cumbersome and wil not be presented. Instead of this, let us denote by $M^{(k)}$ the matrix, obtained after taking the first k matrix - rows and k matrix - columns in the $n-$ dimensional block matrix $M^{(n)}$:

$$M^{(n)} \equiv \begin{pmatrix} E_1^{(1)} & E_1^{(2)} & .... & E_1^{(n)} \\ E_2^{(1)} & E_2^{(2)} & .... & E_2^{(n)} \\ .... & ...... & .... & ...... \\ E_n^{(1)} & E_n^{(2)} & .... & E_n^{(n)} \end{pmatrix} \quad . \tag{C47.}$$

Then one begins first with the calculation of $E_1^{(1)-1}$,then with $M^{(2)}$ (which is in fact the caculated inverse matrix $\widetilde{A}^{-1}$ in (C40), then with $M^{(3)}$ and so on. Each subsequent inverse matrix $M^{(k+1)-1}$ is calculated by applying the Frobenius formulae to the following block matrix:

$$M^{(k+1)} \equiv \begin{pmatrix} M^{(k)} & . & E_1^{(k+1)} \\ & . & E_k^{(k+1)} \\ E_{k+1}^{(1)}...... & E_{k+1}^{(k)} & E_{k+1}^{(k+1)} \end{pmatrix} \quad . \tag{C48.}$$

C3. BLOCK MATRIX STRUCTURE IN THE N-DIMENSIONAL CASE

Following the same algorithm as in the preceeding subsections, we shall try to find the block structure of the system of equations $g_{ij}g^{jk} = \delta_i^k$ in the $n-$dimensional case. For different values of k the system can be written as

$$g^{11}g_{1i} + g^{12}g_{2i} + ... + g^{1n}g_{ni} = \delta_i^1$$

$$g^{21}g_{1i} + g^{22}g_{2i} + ... + g^{2n}g_{ni} = \delta_i^2$$

$$..$$

$$g^{n1}g_{1i} + g^{n2}g_{2i} + ... + g^{nn}g_{ni} = \delta_i^n \tag{C49.}$$

In order to understand the structure of the matrix, it would be useful to write the above system of equations for different values of i.

For $i = 1$ the system (C49) is

$$g^{11}g_{11} + g^{12}g_{21} + ... + g^{1n}g_{ni} = 1$$

$$g^{21}g_{11} + g^{22}g_{21} + ... + g^{2n}g_{n1} = 0$$

$$..$$

$$g^{n1}g_{11} + g^{n2}g_{21} + ... + g^{nn}g_{n1} = 0 \quad , \tag{C50.}$$

or it can be written as

$$AX_1 = \begin{pmatrix} 1 \\ 0 \\ .. \\ 0 \end{pmatrix} \quad , \tag{C51.}$$

where the matrix A is

$$A \equiv \begin{pmatrix} g^{11} & g^{12} & .... & g^{1n} \\ g^{21} & g^{22} & .... & g^{2n} \\ ..... & ..... & .... & ..... \\ g^{n1} & g^{n2} & .... & g^{nn} \end{pmatrix} \tag{C52.}$$

and X_1^T is the transponed vector

$$X_1^T \equiv (g_{11}, g_{21}, ..., g_{n1}) \quad . \tag{C53.}$$

For $i = 2$ the system (C49) can be written as

$$AX_2 = \begin{pmatrix} 0 \\ 1 \\ .. \\ 0 \end{pmatrix} \quad , \tag{C54.}$$

where X_2^T is the transponed vector

$$X_2^T \equiv (g_{12}, g_{22}, ..., g_{2n}) \quad . \tag{C55.}$$

For $i = k$ the system will be $AX_k = \begin{pmatrix} 0 \\ .. \\ 1 \\ ... \end{pmatrix}$, where "1" is on the $k-$th place and the vector X_k^T is

$$X_k^T \equiv (g_{k1}, g_{k2}, ..., g_{kn}) \quad . \tag{C56.}$$

The corresponding vectors $X_1, X_2, ..., X_k$ represent the rows of t he symmetric matrix N

$$N \equiv \begin{pmatrix} g_{11} & g_{12} & ...... & g_{1n} \\ g_{21} & g_{22} & ..... & g_{2n} \\ .... & .... & .... & ... \\ g_{n1} & g_{n2} & .... & g_{nn} \end{pmatrix} \quad , \tag{C57.}$$

in which the unknown variables are in the lower triangular (half) part of the matrix (denoted by N^{tr}).

Let us construct a $\frac{n(n+1)}{2} \times n^2$ dimensional matrix B, which will multiply a $\frac{n(n+1)}{2}$ dimensional vector Y, formed by joining all the consequent rows of the triangular matrix N^{tr}

$$Y \equiv (g_{11}, g_{12}, ..., g_{1n}, g_{22}, g_{23}, ..., g_{2n}, ..., g_{n1}, g_{n2}, ..., g_{nn}) \quad . \tag{C58.}$$

Correspondingly the matrix B will have the **following block triangular structure**:

$$B \equiv \begin{pmatrix} B_{11} & 0 & ...... & 0 \\ B_{21} & B_{22} & ..... & 0 \\ .... & .... & .... & ... \\ B_{n1} & B_{n2} & .... & B_{nn} \end{pmatrix} \quad , \tag{C59.}$$

where each of the block matrices B_{kk} on the diagonal is an $n \times (n-k+1)$ (i.e. $(n-k+1)$ columns and n rows) dimensional matrix, obtained from the matrix A by removing the first $(k-1)$ columns. For example, $B_{11} \equiv A$, but

$$B_{22} \equiv \begin{pmatrix} g^{12} & g^{13} & ...... & g^{1n} \\ g^{22} & g^{23} & ..... & g^{2n} \\ .... & .... & .... & ... \\ g^{n2} & g^{n3} & .... & g^{nn} \end{pmatrix} \quad . \tag{C60.}$$

The corresponding matrix B_{kk} will be

$$B_{kk} \equiv \begin{pmatrix} g^{1k} & g^{1,(k+1)} & .... & g^{1,n} \\ g^{2k} & g^{2,(k+1)} & .... & g^{2,n} \\ ... & .... & .... & ..... \\ g^{nk} & g^{n,(k+1)} & ... & g^{nn} \end{pmatrix} \quad . \tag{C61.}$$

The block - matrices B_{sk} $(s > k)$ are $n \times (n-k+1)$ dimensional ones with just one nonzero column (the $s-k+1$ column) with the elements $g^{k1}, g^{k2}, ...$

$$B_{sk} \equiv \begin{pmatrix} 0 & ... & g^{k1} & ... & 0 \\ 0 & .... & g^{k2} & .... & 0 \\ ... & .... & ... & ... & ... \\ 0 & ... & g^{k,(n-1)} & .... & 0 \\ 0 & ... & g^{k,n} & .... & 0 \end{pmatrix} \quad . \tag{C62.}$$

Let us now have a look at the complex of neighbouring block matrices around the main block diagonal

$$K^{(k)} \equiv \begin{pmatrix} B_{k-1,k-1} & B_{k-1,k} \\ B_{k,k-1} & B_{kk} \end{pmatrix} \quad . \tag{C63.}$$

For illustration of the block matrix multiplication and in order to derive some useful formulaes, let us calculate $K^{(p)T}K^{(k)}$, which will be equal to

$$\begin{pmatrix} B^T_{p-1,p-1}B_{k-1,k-1} + B^T_{p,p-1}B_{k,k-1} & B^T_{p,p-1}B_{k,k} \\ B^T_{pp}B_{k,k-1} & B^T_{pp}B_{kk} \end{pmatrix} \quad . \tag{C64.}$$

The corresponding terms in the above matrix are:

$$B^T_{p-1,p-1}B_{k-1,k-1} =$$

$$= \begin{pmatrix} g^{1,(p-1)} & g^{2,(p-1)} & .... & g^{n,(p-1)} \\ g^{1,p} & g^{2,p} & .... & g^{n,p} \\ ... & .... & .... & ..... \\ g^{1,n} & g^{2,n} & ... & g^{nn} \end{pmatrix} \begin{pmatrix} g^{1,(k-1)} & g^{1,k} & .... & g^{1,n} \\ g^{2,(k-1)} & g^{2,k} & .... & g^{2,n} \\ ... & .... & .... & ..... \\ g^{n,k-1} & g^{n,k} & ... & g^{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} P_{11} & P_{12} & ... & P_{n-k+2} \\ .... & .... & ... & ..... \\ .. & .... & P_{sr.....} & ...... \\ P_{n-p+2,1} & P_{n-p+2,2} & ...... & P_{n-p+2,n-k+2} \end{pmatrix} , \qquad \text{(C65.)}$$

where all the elements of the matrix are nonzero and the element P_{sr} on the $s-$th row and on the $r-$th column is equal to $P_{sr} \equiv \sum_{i=1}^{n} g^{i,p+s-2} g^{i,k+r-2}$,

$$B^T_{p,p-1} B_{k,k-1} \equiv \begin{pmatrix} 0 & 0 & .... & 0 \\ g^{p-1,1} & g^{p-1,2} & ... & g^{p-1,n} \\ ...... & ...... & .... & ..... \\ 0 & 0 & .... & 0 \end{pmatrix} \begin{pmatrix} 0 & g^{k-1,1} & ..... & 0 \\ 0 & g^{k-1,2} & ... & 0 \\ ..... & ...... & .... & ..... \\ 0 & g^{k-1,n} & ... & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 0 & .. & 0 \\ 0 & \sum_{i=1}^{n} g^{(p-1),i} g^{(k-1),i} & ... & 0 \\ ..... & ...... & .... & ... \\ 0 & 0 & ..... & 0 \end{pmatrix} , \qquad \text{(C66.)}$$

where $B^T_{p,p-1}$ and $B_{k,k-1}$ are $(n-p+2) \times n$ and $n \times (n-k+2)$ matrices and the resulting $n \times n$ matrix has only one nonzero element $\sum_{i=1}^{n} g^{(p-1),i} g^{(k-1),i}$ **on the second row and on second column**. Next let us calculate the matrix $B^T_{pp} B_{k,k-1}$, which is a product of the $(n-p+1) \times n$ matrix B^T_{pp} and the $n \times (n-k+2)$ matrix $B_{k,k-1}$ with the only nonzero second column:

$$B^T_{pp} B_{k,k-1} \equiv$$

$$\equiv \begin{pmatrix} g^{1,p} & g^{2,p} & .. & .. & g^{n,p} \\ .... & .... & .. & ... & .... \\ g^{1,p+r-1} & g^{2,p+r-1} & ... & ... & g^{n,p+r-1} \\ .... & ... & .. & .. & .. \\ g^{1,n} & g^{2,n} & .. & .... & g^{nn} \end{pmatrix} \begin{pmatrix} 0 & g^{k-1,1} & ... & 0 & 0 \\ 0 & g^{k-1,2} & ... & 0 & 0 \\ ... & .... & .... & .. & ... \\ 0 & g^{k-1,n-1} & ... & 0 & 0 \\ 0 & g^{k-1,n} & .. & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & F_{12} & ... & 0 \\ 0 & F_{22} & ... & 0 \\ .... & .. & ... & ... \\ 0 & F_{n-p+1,2} & ... & 0 \end{pmatrix} . \qquad \text{(C67.)}$$

The resulting matrix $B^T_{pp} B_{k,k-1}$ has a dimension $(n-p+1) \times (n-k+2)$ with the only nonzero second column with an element on the $r-$th row and on the $2-$nd column $F_{r2} \equiv \sum_{i=1}^{n} g^{i,(p+r-1)} g^{k-1,i}$.

Next let us find the matrix $B^T_{p,p-1}B_{k,k}$, which is a product of the $(n-p+2)\times n$ matrix $B^T_{p,p-1}$ and the $n\times(n-k+1)$ matrix $B_{k,k}$:

$$B^T_{p,p-1}B_{k,k} =$$

$$= \begin{pmatrix} 0 & 0 & ... & 0 \\ g^{(p-1),1} & g^{(p-1),2} & ... & g^{(p-1),n} \\ ... & ... & ... & .... \\ 0 & 0 & .. & 0 \end{pmatrix} \begin{pmatrix} g^{1,k} & .. & g^{1,(k+s-1)} & .. & g^{1,n} \\ g^{2,k} & .. & g^{2,(k+s-1)} & .. & g^{2,n} \\ .. & . & .. & .. & .. \\ ... & .. & ... & ... & .. \\ g^{kk} & g^{k,(k+1)} & ... & .. & g^{kn} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & .. & 0 & .. & 0 \\ \sum_{i=1}^{n} g^{(p-1),i}g^{i,k} & .. & \sum_{i=1}^{n} g^{(p-1),i}g^{i,(k+s-1)} & .. & \sum_{i=1}^{n} g^{(p-1),i}g^{i,n} \\ .. & ... & .... & ... & .. \\ 0 & .. & 0 & .. & 0 \\ 0 & .. & 0 & .. & 0 \end{pmatrix} , \qquad \text{(C68.)}$$

where in the last formulae we have used that the obtained matrix has $(n-k+1)$ columns and therefore the indice $k+s-1$ in the expression for the element $\sum_{i=1}^{n} g^{(p-1),i}g^{i,(k+s-1)}$ on the 2$-$nd row and $s-$th column ranges from k to $k+s-1 = k+(n-k+1)-1 = n$.

It remains only to calculate the matrix $B^T_{pp}B_{kk}$, but it is the same as (C65), this time with an element

$$P_{sr} \equiv \sum_{i=1}^{n} g^{i,p+s-1}g^{i,k+r-1} \qquad \text{(C69.)}$$

on the $s-$th row and on the $r-$th column.

Using the above developed techniques for matrix multiplication, let us calculate the $\frac{n(n+1)}{2}\times\frac{n(n+1)}{2}$ matrix B^TB (recall - B is an $n^2\times\frac{n(n+1)}{2}$ matrix and B^T is an $\frac{n(n+1)}{2}\times n^2$ matrix), which is the $n-$dimensional analogue of the matrix (C19). Using (C63), one has

$$B^TB = \begin{pmatrix} \sum_{i=1}^{n} B^T_{i1}B_{i1} & \sum_{i=2}^{n} B^T_{i1}B_{i2} & ... & \sum_{i=n}^{n} B^T_{i1}B_{in} \\ \sum_{i=2}^{n} B^T_{i2}B_{i1} & \sum_{i=2}^{n} B^T_{i2}B_{i2} & ... & \sum_{i=n}^{n} B^T_{i2}B_{ni} \\ .... & .. & ... & ... \\ \sum_{i=n}^{n} B^T_{in}B_{i1} & .. & ... & \sum_{i=n}^{n} B^T_{in}B_{in} \end{pmatrix} . \qquad \text{(C70.)}$$

The above matrix contains **three types of elements**:

1-st type. Elements below the block diagonal of the type $\sum_{\alpha=j}^{n} B^T_{\alpha j}B_{\alpha k}$ **with** $k < j$**.** Carrying out the matrix multiplication and for the moment not taking the summation

over α, we find

$$B^T_{\alpha j}B_{\alpha k}=\begin{pmatrix} 0 & 0 & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & \sum_{i=1}^{n} g^{ji}g^{ki} & .. & 0 \\ 0 & .. & .. & .. & .0 \\ 0 & 0 & .. & .. & 0 \end{pmatrix}, \qquad \text{(C71.)}$$

where the only nonzero element $\sum_{i=1}^{n} g^{ji}g^{ki}$ in the matrix is on the $(\alpha-j+1)$ row and on the $(\alpha-k+1)$ column and the matrices $B^T_{\alpha j}$ and $B_{\alpha k}$ are of dimensions $(n-j+1)\times n$ and $n\times(n-k+1)$ correspondingly.

Since the first term in the sum $\sum_{\alpha=j}^{n} B^T_{\alpha j}B_{\alpha k}$ will be $B^T_{\alpha\alpha}B_{\alpha k}$, let us find it, performing the same kind of matrix multiplication as in (C67):

$$B^T_{\alpha\alpha}B_{\alpha k}=\begin{pmatrix} 0 & 0 & \sum_{i=1}^{n} g^{i\alpha}g^{ki} & 0 & 0 \\ .. & .. & .. & .. & .. \\ 0 & .. & \sum_{i=1}^{n} g^{i,\alpha+r-1}g^{k,i} & .. & 0 \\ 0 & .. & .. & .. & .0 \\ 0 & 0 & \sum_{i=1}^{n} g^{i,n}g^{k,i} & .. & 0 \end{pmatrix}, \qquad \text{(C72.)}$$

where the only nonzero column is the $(\alpha-k+1)$ one and the element in this column and on the $r-$th row is $\sum_{i=1}^{n} g^{i,\alpha+r-1}g^{k,i}$. The matrices $B^T_{\alpha\alpha}$ and $B_{\alpha k}$ are of dimensions $(n-\alpha+1)\times n$ and $n\times(n-k+1)$ correspondingly and the resulting matrix $B^T_{\alpha\alpha}B_{\alpha k}$ is $(n-\alpha+1)\times(n-k+1)$ dimensional.

2-nd type. Elements above the block diagonal of the type $\sum_{\alpha=k}^{n} B^T_{\alpha j}B_{\alpha k}$ with $k>j$ [The summation indice α takes at first the value of that indice (k or j), which is greater than the other]**.** The first term in the above sum is $B^T_{\alpha j}B_{\alpha\alpha}$, which similarly to the expression (C68) can be found to be

$$B^T_{\alpha j}B_{\alpha\alpha}=$$

$$=\begin{pmatrix} 0 & 0 & .. & .. & 0 \\ .. & .. & .. & .. & .. \\ g^{j1} & g^{j2} & .. & . & g^{jn} \\ .. & .. & .. & .. & .. \\ 0 & 0 & .. & 0 & 0 \end{pmatrix}\begin{pmatrix} g^{1\alpha} & g^{2\alpha} & .. & g^{n\alpha} \\ g^{1,(\alpha+1)} & g^{2,(\alpha+1)} & . & g^{n,(\alpha+1)} \\ ... & ... & .. & .. \\ g^{1,n} & g^{2,n} & ... & g^{n,n} \end{pmatrix}=$$

$$
= \begin{pmatrix} 0 & 0 & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ \sum_{i=1}^{n} g^{j,i} g^{1,(\alpha+i-1)} & ... & \sum_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)} & .. & \sum_{i=1}^{n} g^{j,i} g^{n,(\alpha+i-1)} \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 0 & 0 \end{pmatrix} \quad . \qquad \text{(C73.)}
$$

The above $(n-j+1) \times (n-\alpha+1)$ matrix $B_{\alpha j}^{T} B_{\alpha\alpha}$ contains a nonzero $(\alpha - j + 1)$ row with an element in the $p-$th column, equal to

$$
K_{\alpha-j+1,p} = \sum_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)} \quad . \qquad \text{(C73b.)}
$$

3-rd type. Elements situated on the block diagonal of the type

$$
\sum_{\alpha=k}^{n} B_{\alpha k}^{T} B_{\alpha k} = B_{\alpha\alpha}^{T} B_{\alpha\alpha} + \sum_{\alpha=k+1}^{n} B_{\alpha k}^{T} B_{\alpha k} \quad . \qquad \text{(C74.)}
$$

Similarly to the calculation of (C69), the first term in (C74) ($B_{\alpha\alpha}^{T}$ is an $(n-\alpha+1) \times n$ matrix ; $B_{\alpha\alpha}$ is an $n \times (n-\alpha+1)$ matrix) is found to be the following $(n-\alpha+1) \times (n-\alpha+1)$ matrix (also, $\alpha = k$) :

$$
B_{\alpha\alpha}^{T} B_{\alpha\alpha} = \begin{pmatrix} N_{11} & N_{12} & ... & N_{1,(n-\alpha+1)} \\ N_{21} & N_{22} & ... & N_{2,(n-\alpha+1)} \\ ... & ... & ... & .. \\ N_{n-\alpha+1,1} & ... & ... & N_{n-\alpha+1,n-\alpha+1} \end{pmatrix} \quad , \qquad \text{(C75.)}
$$

where

$$
N_{pq} = \sum_{i=1}^{n} g^{i,\alpha+p-1} g^{i,\alpha+q-1} \quad . \qquad \text{(C76.)}
$$

The second term in (C74) is in fact the $(n-k+1) \times (n-k+1)$ matrix (C71) for $j = k$. The summation over α from $\alpha = k+1$ to n will give a diagonal $(n-k+1) \times (n-k+1)$ matrix with an element $G_{\alpha-k+1,\alpha-k+1} = \sum_{i=1}^{n} (g^{k,i})^2$ on the $(\alpha - k + 1)$ row and on the $\alpha - k + 1)$ column, which will be situated on the main (**block)** diagonal from $\alpha = k+1$ to n. Since for $\alpha = k+1$ we have $\alpha - k + 1 = k + 1 - k + 1 = 2$ and for $\alpha = n$ we have also $\alpha - k + 1 = n - k + 1$, this means that the matrix $\sum_{\alpha=k+1}^{n} B_{\alpha k}^{T} B_{\alpha k}$ will have the following structure:

$$
\sum_{\alpha=k+1}^{n} B_{\alpha k}^{T} B_{\alpha k} = \begin{pmatrix} 0 & 0 & .. & 0 & 0 \\ 0 & \sum_{i=1}^{n} (g^{k,i})^2 & .. & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{n} (g^{k,i})^2 & .. & 0 \\ ... & ... & .... & .... & ... \\ 0 & 0 & .... & 0 & \sum_{i=1}^{n} (g^{k,i})^2 \end{pmatrix} \quad . \qquad \text{(C77.)}
$$

Therefore, summing up the matrices (C75) and (C77), one obtains **the general structure of the matrices (C74)** $\sum\limits_{\alpha=k}^{n} B_{\alpha k}^T B_{\alpha k}$ **on the block diagonal:**

$$\sum_{\alpha=k}^{n} B_{\alpha k}^T B_{\alpha k} = \begin{pmatrix} N_{11} & N_{12} & ... & N_{1,(n-\alpha+1)} \\ N_{21} & N_{22}+G_{22} & ... & N_{2,(n-\alpha+1)} \\ ... & ... & ... & .. \\ N_{n-\alpha+1,1} & ... & ... & N_{n-\alpha+1,n-\alpha+1}+G_{22} \end{pmatrix} \quad , \qquad \text{(C78.)}$$

where N_{pq} and G_{22} are given by expressions (C76) and $G_{2,2} = \sum\limits_{i=1}^{n}(g^{k,i})^2$ respectively. Consequently (for $p \geq 2$)

$$N_{pp} + G_{22} = \sum_{i=1}^{n} \left[g^{i,(k+p-1)} g^{i,(k+p-1)} + (g^{k,i})^2 \right] \qquad . \qquad \text{(C79.)}$$

Let us find now the structure of the **off - diagonal block matrices**, situated **below the diagonal in the block matrice (C70)**. Each such an element can be decomposed as

$$\sum_{\alpha=j(j>k)} B_{\alpha j}^T B_{\alpha k} = B_{\alpha\alpha}^T B_{\alpha k} + \sum_{\alpha=j+1}^{n} B_{\alpha j}^T B_{\alpha k} \qquad . \qquad \text{(C80.)}$$

The first term in (C80) is the already known $(n-\alpha+1)\times(n-\alpha+1)$ matrix (C72) with the only nonzero $(j-k+1)$ column. The second term is the sum from $\alpha = j+1$ to $\alpha = n$ of the $(n-\alpha+1)\times(n-k+1)$ matrices $B_{\alpha j}^T B_{\alpha k}$, already calculated in (C71) and having the only nonzero element $\sum\limits_{i=1}^{n} g^{ji} g^{ki}$ on the $(\alpha-j+1)$ row and on the $(\alpha-k+1)$ column. The summation over α from $\alpha = j+1$ to $\alpha = n$ means that in the sum $\sum\limits_{\alpha=j+1}^{n} B_{\alpha j}^T B_{\alpha k}$ the element $\sum\limits_{i=1}^{n} g^{ji} g^{ki}$ will appear beginning from the row $\alpha-j+1 = 2$ (for $\alpha = j+1$) up to the row $\alpha-j+1 = n-j+1$ (for $\alpha = n$), which in fact is the **last row.** Correspondingly, the same element will appear beginning from the column $\alpha-k+1 = j-k+2$ (for $\alpha = j+1$) and ending at the column $\alpha-k+1 = n-k+1$ (at $\alpha = n$), which is **the last column**. In other words, the summation of the matrices $B_{\alpha j}^T B_{\alpha k}$,containing one element, effectively results in a matrix, filled up from the second row to the end and from the $(j-k+2)$ column to the end:

$$\sum_{\alpha=j+1}^{n} B_{\alpha j}^T B_{\alpha k} = \sum_{\alpha=j+1}^{n} \begin{pmatrix} 0 & .. & 0 & .. & 0 \\ ... & .. & .... & ... & ... \\ 0 & .... & \sum\limits_{i=1}^{n} g^{ji} g^{ki} & .... & 0 \\ ... & ... & ... & ... & ... \\ 0 & .. & 0 & ... & 0 \end{pmatrix} =$$

$$
= \begin{pmatrix} 0 & ... & \text{column } (j-k+2)... & 0 & 0 \\ 0 & ... & \sum_{i=1}^{n} g^{ji}g^{ki} & .. & \sum_{i=1}^{n} g^{ji}g^{ki} \\ .. & .. & .. & .. & .. \\ 0 & .. & \sum_{i=1}^{n} g^{ji}g^{ki} & .. & \sum_{i=1}^{n} g^{ji}g^{ki} \\ 0 & .. & \sum_{i=1}^{n} g^{ji}g^{ki} & .. & \sum_{i=1}^{n} g^{ji}g^{ki} \end{pmatrix} . \qquad \text{(C81.)}
$$

Now recall that the matrix $B^T_{\alpha\alpha}B_{\alpha k}$ (C72) had a nonzero $(j-k+1)$ column, so therefore the structure of the whole matrix $\sum\limits_{\alpha=j}^{n} B^T_{\alpha j}B_{\alpha k}$ $(j>k)$ **below the diagonal** is similar to (C81), but with the additional $(j-k+1)$ column:

$$
\begin{pmatrix} 0 & 0.......... & \sum_{i=1}^{n} g^{i,\alpha}g^{k,i} & 0 & .. & 0 \\ 0 & 0...... & \sum_{i=1}^{n} g^{i,\alpha+1}g^{k,i} & \sum_{i=1}^{n} g^{ji}g^{ki} & ... & \sum_{i=1}^{n} g^{ji}g^{ki} \\ ... & ... & ..... & ... & ... & ... \\ 0 & 0........ & \sum_{i=1}^{n} g^{i,\alpha+r-1}g^{k,i} & \sum_{i=1}^{n} g^{ji}g^{ki} & .. & \sum_{i=1}^{n} g^{ji}g^{ki} \\ .. & ..... & .. & .. & ... & .. \\ 0 & 0..... & \sum_{i=1}^{n} g^{i,n}g^{k,i} & \sum_{i=1}^{n} g^{ji}g^{ki} & .. & \sum_{i=1}^{n} g^{ji}g^{ki} \end{pmatrix} . \qquad \text{(C82a.)}
$$

Since the structure of this matrix is important, let us write it in a more precise way:

$$
\sum_{\alpha=j}^{n} B^T_{\alpha j}B_{\alpha k} \equiv \begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \end{pmatrix} \quad , \qquad \text{(C82b.)}
$$

where M_1, M_3 and M_4 are zero matrices of dimensions $1\times(j-k)$, $1\times l$ and $(n-j)\times(j-k)$ respectively and the number l is equal to $l = n-k+1-(j-k+1)+1 = n-j+1$. The matrix M_2 is 1×1 and contains the only element $\sum\limits_{i=1}^{n} g^{i,\alpha}g^{k,i}$, M_5 is $(n-j)\times 1$ dimensional and is therefore the column

$$
M_5 \equiv \begin{pmatrix} \sum_{i=1}^{n} g^{i,\alpha+1}g^{k,i} \\ ... \\ \sum_{i=1}^{n} g^{i,\alpha+r-1}g^{k,i} \\ .... \\ \sum_{i=1}^{n} g^{i,n}g^{k,i} \end{pmatrix} \qquad \text{(C82c.)}
$$

and M_6 is a $(n-j)\times l$ matrix, in which each element is equal to $\sum\limits_{i=1}^{n} g^{ji}g^{ki}$.

In a completely analogous way the elements **above the block diagonal** in (C70) can be found. These elements $\sum\limits_{\alpha=k}^{n} B_{\alpha j}^{T} B_{\alpha k}$ $(k > j)$ can be decomposed as

$$\sum_{\alpha=k(k>j)} B_{\alpha j}^{T} B_{\alpha k} = B_{kj}^{T} B_{kk} + \sum_{\alpha=k+1}^{n} B_{\alpha j}^{T} B_{\alpha k} \quad . \tag{C83.}$$

This formulae is similar to (C80) for the below - diagonal elements, but here in (C83) we have the matrix $B_{kj}^{T} B_{kk}$ instead of the matrix $B_{\alpha\alpha}^{T} B_{\alpha k}$ and the summation over the indice α in the second term is from $\alpha = k+1$ to $\alpha = n$ instead of $\alpha = j+1$ to $\alpha = n$ in (C80). The first term in (C83) $B_{kj}^{T} B_{kk}$ is the $(n-j+1) \times (n-k+1)$ matrix (C73) (with $\alpha = k$) with the only nonzero $(k-j+1)$ row with the elements

$$\sum_{i=1}^{n} g^{ji} g^{1,(k+i-1)} \quad ; \dots . \sum_{i=1}^{n} g^{ji} g^{p,(k+i-1)} ; \dots .. \sum_{i=1}^{n} g^{ji} g^{n,(k+i-1)} \tag{C84.}$$

The second term in (C83) is again the sum of the matrices (C71). The only nonzero element $\sum\limits_{i=1}^{n} g^{ji} g^{ki}$ will now appear in the final sum from the $\alpha - j + 1 = k - j + 2$ row $(\alpha = k+1)$ until the $k - j + 1 = n - j + 1$ $(\alpha = n)$ row, which is the **last one**. Also, the same element will fill up the columns from $\alpha - k + 1 = 2$ $(\alpha = k+1)$ to the column $\alpha - k + 1 = n - k + 1$ $(\alpha = n)$, which is also the **last one**. Therefore, summing up the two terms in (C83), one obtains the following matrix for the **above - diagonal block terms,** in which the $(k-j+1)$ row is filled up with the elements $\sum\limits_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)}$ and from the next $(k-j+2)$ row to the end and from the second column to the end column the matrix is filled up with the other element $\sum\limits_{i=1}^{n} g^{ji} g^{ki}$:

$$\begin{pmatrix} 0 & 0\dots\dots\dots & 0.. & 0.. & .. & 0 \\ .. & \dots\dots & \dots & \dots & \dots & \dots. \\ \sum\limits_{i=1}^{n} g^{j,i} g^{1,(\alpha+i-1)} & \sum\limits_{i=1}^{n} g^{j,i} g^{2,(\alpha+i-1)} & \dots\dots\dots \sum\limits_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)} & \dots & \dots & \sum\limits_{i=1}^{n} g^{j,i} g^{n,(\alpha+i-1)} \\ 0 & \sum\limits_{i=1}^{n} g^{ji} g^{ki} & \dots.. & \sum\limits_{i=1}^{n} g^{ji} g^{ki} & .. & \sum\limits_{i=1}^{n} g^{ji} g^{ki} \\ .. & \dots.. & .. & .. & \dots & .. \\ 0 & \sum\limits_{i=1}^{n} g^{ji} g^{ki} & \sum\limits_{i=1}^{n} g^{i,n} g^{k,i} & \sum\limits_{i=1}^{n} g^{ji} g^{ki} & .. & \sum\limits_{i=1}^{n} g^{ji} g^{ki} \end{pmatrix} . \tag{C85.}$$

Let us now summarize the obtained results for the $n-$dimensional case. The (predetermined) system of equations $g_{ij} g^{jk} = \delta_{i}^{k}$ was represented as $BY = \overline{1}$, where $\overline{1}$ is an n^2 dimensional vector, whose transponed one is defined as $\overline{1}^{T} = (\overline{1}^{T1}, \overline{1}^{T2}, \dots\dots\dots\dots, \overline{1}^{Tn})$ and the corresponding transponed $n-$ dimensional vectors $\overline{1}^{T1}, \overline{1}^{T2}, \dots\dots\dots\dots, \overline{1}^{Tn}$ are defined

as follows: $\overline{1}^{T1} = (1, 0, ..., 0), \overline{1}^{T2} = (0, 1, 0, ...0)$ and the $k-$th transposed vector $\overline{1}^{Tk} = (0, 0, ..., 1, 0, ...0)$ contains the number 1 on the $k-$th place. The $n^2 \times \frac{n(n+1)}{2}$ matrix B in terms of the elementary block matrices B_{ij} and the $\frac{n(n+1)}{2}$ dimensional vector Y were defined by formulaes (C59 - C62) and (C58) respectively. In order to solve the system, we multiplied it to the left with the transponed matrix B^T and thus the solution for the vector Y in matrix notations can be found as $Y = (B^T B)^{-1} B^T \overline{1}$, where the expression for $B^T \overline{1}$ can easily be found to be

$$B^T \overline{1} = \begin{pmatrix} B_{11}^T & B_{21}^T & ...... & B_{n1}^T \\ 0 & B_{22}^T & .... & B_{n2}^T \\ ... & ... & ... & .. \\ 0 & 0 & .. & B_{nn}^T \end{pmatrix} \begin{pmatrix} \overline{1}^1 \\ \overline{1}^2 \\ ... \\ \overline{1}^m \end{pmatrix} = \begin{pmatrix} \sum\limits_{i=1}^{n} B_{i1}^T \overline{1}^i \\ \sum\limits_{i=1}^{n} B_{i2}^T \overline{1}^i \\ ... \\ \sum\limits_{i=1}^{n} B_{im}^T \overline{1}^i \end{pmatrix} \quad . \qquad \text{(C86.)}$$

The expressions for the elements of the vector in the R. H. S. can also be found, but this will not be performed here and will be left for the interested reader. Let us remind again that each "element" of this vector is in itself an $n-$dimensional vector and $m = \frac{n(n+1)}{2}$ gives the number of the "block" elements of this vector - column. The number m should be an integer number, but this requirement can be fulfilled and this will be commented in the next subsection.

The main result of this subsection are contained in the expressions (C78), (C82a) and (C85) for the elements of the matrix $B^T B$ on the block diagonal, below the block diagonal and above the block diagonal respectively. In such a way the detailed structure of the matrtix $B^T B$ is known in terms of the elementary constituent block matrices B_{ij}. The structure of the matrix $B^T B$ is important for the following reasons:

1. If one chooses the contravariant metric tensor in the form of a factorized product $\widetilde{g}^{ij} = dX^i dX^j$, then one can answer the question whether it is possible the matrix $B^T B$ to have a lower rank and some common multiplier. But from the above expressions and since in each "cell" of the elementary block matrices one has a summation of the kind $\sum\limits_{i=1}^{n} g^{i,n} g^{k,i}$, it is evident that such a common multiplier does not exist and therefore the rank of the matrix $B^T B$ (and therefore of B) cannot be lowered due to the above made choice of g^{ij}.

2. The quadratic structure of the matrix $B^T B$ gives an opportunity to apply the Frobenius formulae for finding the inverse matrix $(B^T B)^{-1}$ in the sense, in which this was discussed at the end of the previous subsection. However, the application of the Frobenius foemulaes presumes that the inverse matrix of the upper left "element" $\sum\limits_{i=1}^{n} B_{i1}^T B_{i1}$ in the matrix (C70) exits (which probably is the case due to the expression (C78) for the block diagonal elements), and the subsequent application of the Frobenius formulae to the 2×2 block matrix "complex" in (C70) will also give an invertible matrix, then also with the

3×3 complex and so on. Probably it is interesting to derive a recurrent formulae for $(B^T B)^{-1}$, depending on the elementary constituent block matrices. In any case, if they are invertible matrices, this shall turn out to be important, because it will be shown further how the matrix B may be "divided" into $n \times n$ block matrices, which **might not** be invertible.

C4. BLOCK MATRIX REPRESENTATION OF THE HOMOGENEOUS SYSTEM OF EQUATIONS WITH A ZERO R. H. S.

Earlier it was shown that for the case $n = 3$ the sub - system of equations $g_{ij} g^{jk} = 0$ with a zero R. H. S. has a determinant of coefficient (functions), equal to zero. **The question which naturally arises is whether this property is valid only for the $n = 3$ case and is it valid for the $n-$ dimensional case.**

Below it shall be proved that this can be done for the general case. Namely, it shall be established that from the $n^2 - n$ equations with a zero R. H. S. there may be chosen a sub-system of $\left(\begin{array}{c} n \\ 2 \end{array} \right) + n = \frac{n(n+1)}{2}$ equations with a determinant of coefficients, equal to zero. It shall be stressed that the proof will be that **such a system exists** (i.e. can be chosen) and **not that all other choices of the subsystem of equations will also satisfy this requirement.** In fact, investigating under other choices of the sub - system of equations with a zero R. H. S. this property will be preserved represents an interesting problem for further research.

For the purpose, let us again use the block matrix representation (C59). Then, in order to obtain the system of equations with a zero R. H. S. , one has to remove the first row (for $i = k = 1$) from the system (C50) of the first n equations, then the second row from the second system of n equations and so on, one has to remove the $k-$th row from the $k-$th system of n equations. Further, in order to receive again a block matrix structure with (elementary) submatrices with n rows, one should **add the first row of the second system** of n equations as the **last row of the first system** of equations. In effect, since the **first two rows of the second system have been removed**, one should add **the first two rows of the third system** of n equations as **the last two rows of the second system**. Therefore, since also the third row in the third system (for $i = k = 3$) has been removed, it has a total of three rows removed, and subsequently **three last rows** have to be added from the fourth system. Continuing in the same manner, from the $k-$th matrix B_{kk} on the block diagonal we have the first $k-$ rows from B_{kk} removed and also k - last rows added, which should be taken from the below - diagonal matrix $B_{k+1,k}$. Since according to (C62) the $n \times (n - k + 1)$ matrix B_{sk} has a nonzero $(s - k + 1)$ column, the matrix $B_{k+1,k}$ will have a nonzero $k + 1 - k + 1 = 2$ column. Therefore the new transformed in this way $n \times (n - k + 1)$ matrix, which will be denoted as $\widetilde{B}_{kk}$, will

have the following structure:

$$\widetilde{B}_{kk} \equiv \begin{pmatrix} g^{k+1,k} & g^{k+1,k+1} & .. & .. & ... & .. & g^{k+1,n} \\ g^{k+2,k} & g^{k+2,k+1} & .. & .. & . & . & g^{k+2,n} \\ .. & .. & .. & .. & . & .. & ... \\ g^{nk} & g^{n,(k+1)} & .. & .. & .. & .. & g^{nn} \\ 0 & g^{k1} & .. & 0 & .. & 0 & 0 \\ .. & & .. & .. & .. & .. & .. \\ 0 & g^{kk} & .. & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{(C87.)}$$

In the same way, the below - diagonal transformed matrix $\widetilde{B}_{k+1,k}$, obtained from $B_{k+1,k}$ by removing its first k and adding k rows from $B_{k+2,k}$, will be of the following kind:

$$\widetilde{B}_{k+1,k} \equiv \begin{pmatrix} 0 & g^{k,(k+1)} & 0 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & g^{k,n} & 0 & .. & 0 & 0 \\ 0 & 0 & g^{k,1} & .. & 0 & 0 \\ .. & .. & .. & ... & .. & .. \\ 0 & 0 & g^{k,k} & .. & 0 & 0 \end{pmatrix} \quad . \qquad \text{(C88.)}$$

Since the block matrix (C59) has a **triangular structure,** for our further purposes only the structure of the block - diagonal matrices $\widetilde{B}_{kk}$ will be relevent.

Next, our goal will be to divide the the block - matrix (C59) into elemetary block matrices with an **equal number (n)** of rows and columns. Let us remind once again that the block - matrice (C59) contained elementary block - matrices with an **unequal number of columns** - $n, (n-1), (n-2)$......correspondingly. For the purpose, we shall take one (left) column from the block matrix with $(n-2)$ columns and transfer it to the left to the block matrice with $(n-1)$ columns. As a result, the block matrices on the second block matrix column (B. M. C.) will already contain n columns. Since in the block matrix column one column has been transfered, one has to add 3 columns from the $(n-3)-$rd block matrix column to the $(n-2)-$nd block matrix column in order to obtain again a block matrix column, consisting of elementary $n \times n$ matrices. Continuing in the same manner with the $(n-3)-$rd B. M. C., one has to add to its right end 6 columns from the $(n-4)-$th B. M. C. Now let us write down the numbers of the corresponding block columns and below with a $(-)$ sign the number of columns, transferred to the (neighbouring) B. M. C. ; with a $(+)$ sign the number of columns, joined to the B. M.C. (to the right side) from the neighbouring (right) B. M. C.

$$\begin{array}{ccccccccc} (n-1) & ; & (n-2) & ; & (n-3) \; ; & & (n-4) & ; & (n-5) \\ +1; & & -1 \quad +3 & ; & -3 \quad +6 & ; & -6 \quad +10 & ; & -10 \quad +15 \end{array} \quad . \qquad \text{(C89.)}$$

Now let us look at the numbers with a minus sign, which form the following number sequence (with the corresponding number in the sequence denoted):

$$1, 3, 6, 10, 15, 21......$$

$$1, 2, 3, 4, 5, 6, ... \qquad \text{(C90.)}$$

It is trivial to note that each number in the sequence (upper row) is in fact a sum of the corresponding numbers in the lower row up to that number. For example, the number 21 in the sequence (upper row) can be represented as a sum of the numbers in the sequence (lower row): $21 = 1 + 2 + 3 + 4 + 5 + 6$. The same with the number 10 $\Longrightarrow 10 = 1+2+3+4$. Therefore, to the sequence number k in the low row will correspond

the number $\frac{k(k+1)}{2}$ in the upper row, which is the sum of the first k numbers in the low row. Since the number k in the lower (C90) corresponds to the $(n-k)-$th block column, the corresponding number will be $\frac{k(k-1)}{2}$, and it will correspond to the number of columns, which have to be removed from the diagonal block matrix $B_{k+1,(k+1)}$. At the same time, to the right end one should add $k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2}$ columns. This number is exactly equal to the number of left columns, removed from the (right) neighbouring matrix $B_{(k+1),(k+2)}$. This can serve also as a consistency check that the performed calculation is consistent and correct.

From the matrix (C87) for $\widetilde{B}_{k+1,(k+1)}$ we have to delete the first $\overline{s}+1$ left columns, the first (upper) elements of which begin with the elements $g^{(k+2),(k+1)}; g^{(k+2),(k+2)}............, g^{(k+2),(\overline{s}+k)}$, where $\overline{s} = \frac{k(k-1)}{2}$. Now let us denote by $\overline{B}_{(k+1),(k+1)}$ the matrix $\widetilde{B}_{k+1,(k+1)}$ with $\frac{k(k+1)}{2}$ left columns removed and $\frac{k(k+1)}{2}$ right columns added. The upper elements of the (left) remaining columns will be $g^{(k+2),(\overline{s}+k+1)}, g^{(k+2),(\overline{s}+k+2)}, ..., g^{(k+2),n}$, where

$$\overline{s} + k + 1 = \frac{k(k-1)}{2} + k + 1 = \frac{k(k+1)}{2} + 1 \qquad \text{(C91.)}$$

and the matrix $\overline{B}_{(k+1),(k+1)}$ will contain $n - \frac{k(k+1)}{2}$ left nonzero columns. Therefore the $n \times n$ matrix $\overline{B}_{(k+1),(k+1)}$ will have the following structure:

$$\overline{B}_{(k+1),(k+1)} \equiv \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \quad , \qquad \text{(C92.)}$$

where the $(n-k+1) \times \left[n - \frac{k(k+1)}{2}\right]$ matrix L_1 is

$$L_1 \equiv \begin{pmatrix} g^{(k+2),(\frac{k(k+1)}{2}+1)} & ... & ... & g^{(k+2),n} \\ g^{(k+3),(\frac{k(k+1)}{2}+1)} & ... & .... & g^{(k+3),n} \\ .... & ... & ... & ... \\ g^{n,\frac{k(k+1)}{2}+1} & ... & ... & g^{n,n} \end{pmatrix} \qquad \text{(C93.)}$$

and L_2, L_3 and L_4 are zero matrices of dimensions $(n-k+1) \times \left[\frac{k(k-1)}{2}\right], (k+1) \times \left[n - \frac{k(k+1)}{2}\right]$ and $(k+1) \times \left[\frac{k(k+1)}{2}\right]$ correspondingly. Note also that the block matrices

L_1 and L_3 contain $\left[n - \frac{k(k+1)}{2}\right]$ nonzero columns, and the remaining $\left[\frac{k(k+1)}{2}\right]$ columns of the matrices L_2 and L_4 are exactly equal to the number of zero columns, added to the right side of the matrix $\overline{B}_{(k+1),(k+1)}$ from the neighbouring matrix $\overline{B}_{(k+1),(k+2)}$. Since this result depends on the initial structure of the matrix $B_{(k+1),(k+1)}$ and on the expression (C91) (which are both independent on the number of removed right columns), this also confirms the consistency of the calculation.

Note that the block structure of the matrix (C92) has been revealed on the base of the assumption that the elements $g^{(k+1),1}, ...g^{(k+1),(k+1)}$ in the last $(k+1)$ rows and the second column in the matrix (C87) will be among the first removed to the left (and outside the matrix) columns. However, for $\overline{s} = 0$ (i.e. $k = 1$) the elements in the second column of the last two rows of the matrix $\overline{B}_{2,2}$ will contain the elements g^{21}, g^{22}. Therefore, after removing the first column in the matrix (C87) $\widetilde{B}_{(k+1),(k+1)}$ (for $k = 1$) and adding to the right one zero column, the obtained structure of the $n \times n$ matrix $\overline{B}_{2,2}$ will be the following:

$$\overline{B}_{2,2} = \begin{pmatrix} g^{33} & g^{34} & ... & g^{3n} & 0 & 0 \\ g^{43} & g^{44} & ... & g^{4n} & 0 & 0 \\ .. & ... & .. & ... & .. & .. \\ g^{n3} & g^{n4} & ... & g^{nn} & 0 & 0 \\ g^{21} & 0 & .. & 0 & 0 & 0 \\ g^{22} & 0 & ... & 0 & 0 & 0 \end{pmatrix} \quad . \qquad \text{(C94.)}$$

Having established the block structure of the matrix of coefficients of the $n-$dimensional predetermined system $g_{ij}g^{jk} = 0$ with a homogeneous zero R. H. S., it is now easy to show that $\frac{n(n+1)}{2}$ equations can be chosen so that the determinant of coefficients will be zero. Let us take for example the first $\frac{n(n+1)}{2}$ equations from the system of $(n^2 - n)$ equations with a zero R. H. S. with the corresponding $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ dimensional block matrix. In the particular case the dimension of the block matrix is determined by the number of block matrices on the horizontal $(\frac{n(n+1)}{2})$t and on t he vertical $(\frac{n(n+1)}{2})$. Therefore, outside this matrix will remain a matrix of $\frac{n(n+1)}{2}$ matrix block columns and $(n-1) - \frac{n(n+1)}{2} = \frac{(n-3)}{2}$ block rows. The last assumption presumes that the spacetime dimension number n is an **odd** one, so that $(n + 1)$ and $(n - 3)$ are dividable by two. Otherwise, if n is an **even number, one may consider** $\frac{n}{2} \times \frac{n}{2}$ dimensional elementary block matrices $\overline{B}_{k,k}$. Then the full block matrix of the system will have $(n + 1)$ block matrices on the block horizontal (i.e. $(n + 1)$ block columns) and $2n^2$ matrices on the block vertical (i. e. $2n^2$ block rows). The block matrix of the homogeneous system of equations (with a zero R. H. S.) will be $2(n - 1) \times (n + 1)$ "block" dimkensional. The chosen block matrix will be $(n + 1) \times (n + 1)$ "block" dimensional. Outside this matrix there will remain a block matrix of $2n - 2 - n - 1 = n - 3$ block rows and $(n + 1)$ block columns.

Let us now compute the determinant of the triangular matrix (C59), from which we take the first $\frac{(n+1)}{2}$ (or $\frac{n}{2}$) block rows. This $\frac{(n+1)}{2} \times \frac{(n+1)}{2}$ (or $\frac{n}{2} \times \frac{n}{2}$) block determinant $\widetilde{S}$

will be equal to

$$\widetilde{S} = \prod_{i=1}^{\frac{n+1}{2}} (det\overline{B}_{ii}) \quad . \tag{C95.}$$

But for $i \neq 1, 2$ the expression for $det\overline{B}_{ii}$ has to be found from formulae (C91) for the block matrix $\overline{B}_{kk}$. Since only one of the submatrices L_1 is different from zero, it is clear that $det\overline{B}_{ii} = detL_1.0 = 0$, therefore the whole expression (C95) equals zero.

Thus we have proved that by taking $\frac{(n+1)}{2}$ (or $\frac{n}{2}$) consequent block rows from the initial (quadratic and triangular) block matrix, the obtained block matrix will have a zero determinant.

14 APPENDIX D: ANALYTICAL CALCULATION OF INTEGRALS $I_{N-1} \equiv \int e^{2\overline{k}\varepsilon y} y^{n-1} dy$

Performing a simple integration by parts

$$I_{n-1} = \frac{\varepsilon}{2k} \int y^{n-1} d(e^{2\overline{k}\varepsilon y}) \quad , \tag{D1}$$

it is easy to find that

$$I_{n-1} = \frac{\varepsilon}{2\overline{k}} y^{n-1} e^{2\overline{k}\varepsilon y} - \left(\frac{\varepsilon}{2\overline{k}}\right)^2 (n-1) y^{n-2} e^{2\overline{k}\varepsilon y} +$$

$$+ \left(\frac{\varepsilon}{2\overline{k}}\right)^2 (n-1)(n-2) I_{n-3} \quad . \tag{D2}$$

Expressing further I_{n-3},..in the same way, the formulae can be generalized as

$$I_{n-1} = \frac{\varepsilon}{2k} y^{n-1} e^{2\overline{k}\varepsilon y} + \sum_{k=1}^{s} (-1)^k \left(\frac{\varepsilon}{2\overline{k}}\right)^2 (n-k) y^{n-k-1} e^{2\overline{k}\varepsilon y} +$$

$$+(-1)^{s+1} (n-1)(n-2)...(n-s-1) I_{n-s-2} \quad . \tag{D3}$$

Let us assume that the final term will be

$$I_{n-s-2} = I_1 = \frac{\varepsilon}{2\overline{k}} y e^{2\overline{k}\varepsilon y} - \left(\frac{\varepsilon}{2\overline{k}}\right)^2 e^{2\overline{k}\varepsilon y} \quad , \tag{D4}$$

which means that

$$n - s - 2 = 1 \implies s = n - 3 \tag{D5}$$

and therefore formulae (D3) acquires the form

$$I_{n-1} = \sum_{k=1}^{n-3} (-1)^k \left(\frac{\varepsilon}{2\overline{k}}\right)^2 (n-k) y^{n-k-1} e^{2\overline{k}\varepsilon y} +$$

$$+\frac{\varepsilon}{2\overline{k}} y^{n-1} e^{2\overline{k}\varepsilon y} + (-1)^{n-2}(n-1)! I_1 \quad . \tag{D6}$$

Now it remains only to calculate the first term with the sum. Let us denote this term by $\widetilde{I}_{n-1}$. If we differentiate this term by y, we may obtain

$$\frac{\partial \widetilde{I}_{n-1}}{\partial y} = 2k\varepsilon \widetilde{I}_{n-1} + \widetilde{I}_{n-2} \quad , \tag{D7}$$

where in general $\widetilde{I}_{n-s}$ will denote

$$\widetilde{I}_{n-s} \equiv \sum_{k=1}^{n-3} (-1)^k \left(\frac{\varepsilon}{2\overline{k}}\right)^2 (n-k)(n-k-1)....(n-k-s+1)\ y^{n-k-s} e^{2\overline{k}\varepsilon y} \quad . \tag{D8}$$

Note that the same equality as (D7) holds for arbitrary s, i.e.

$$\frac{\partial \widetilde{I}_{n-s}}{\partial y} = 2k\varepsilon \widetilde{I}_{n-s} + \widetilde{I}_{n-s-1} \quad . \tag{D9}$$

From this recurrent differential equality it is evident that if one knows $\widetilde{I}_1$, then $\widetilde{I}_2$ can be found as a solution of an diffrential equation; in the same way, if $\widetilde{I}_2$ is known, $\widetilde{I}_3$ can be found and etc. Finally, $\widetilde{I}_{n-1}$ can be found. Of course, one can try a certain starting "probe" function in the form of a polynomial for $\widetilde{I}_1$ with unknown coefficients and then try to find the recurrent relations for these coefficients, so that (D9) is fulfilled.

This purely technical calculations shall not be performed here, because the final answer for the integral $I_{n-1} = \int e^{2\overline{k}\varepsilon y} y^{n-1}$ is already known and can be found too in the monograph by Timofeev [75]

$$I_{n-1} = \frac{e^{2\overline{k}\varepsilon\ y}}{(2\overline{k}\varepsilon)^n} \sum_{p=0}^{n-1} (-1)^p \begin{pmatrix} n-1 \\ p \end{pmatrix} p!(2\overline{k}\varepsilon)^{n-1-p} y^{n-1-p} \quad . \tag{D10}$$

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